Playing with Christoffel sequences in Escher's garden.

Abstract

This paper is autobiographical. The author first expresses his affinity with the work of the Dutch graficus Maurits Escher and the long, lonely journey of this artist through the symmetrical universe, seeking to unveil its secrets. An universe that Escher called 'his garden'. The author for many decades felt himself as wandering through that garden, gradually coming under the spell of quite another kind of beauty as Escher was gripped by there, namely the beauty of Christoffel sequences. Besides the symmetrical operators translation, rotation, reflection and slide reflection by which Escher was fascinated life long, also Christoffel sequences are inextricably linked with the 17 plane symmetries. The mathematical beauty, elegance and magic of Christoffel sequences is described, with reference to the past and current conceptualizations in mathematics. Number theory and discrete mathematics are the disciplines which are shedding light on these sequences. By the way some critical notes are made: Does modern mathematics justice to the beauty, elegance and magic of these sequences, in its longing for quite universal explanation models?

Next is shown how these sequences can be evoked in plane symmetries by introducing a direction there. First this is treated in a more technical sense. Then we shown how beauty and elegance can be genereated in this way . Wonderful spirals and crystals will pass in review.

Content

1Escher's garden

As a child I was deeply interested in the products of frosty weather: frozen sprout stalks in the snowy kitchen garden, icicles hanging from the gutter, frozen puddles in the country-road behind my father's farm, the footprints of a hare in the snow and snowflakes on the window-sill. When growing older it became the principles of order and regularity in general, not necessarily in relation to frost, by which I was highly intrigued. I searched around in the field of crystallography and found out how I could cause crystals to grow. I acquainted myself with the theory of mathematical groups (Speiser) and learned that there are 17 different types of symmetrical arrangement in a plane (Figure 1). After some time my fascination became so strong that I began to perceive principles of arrangement everywhere around me, in the streets, on elevator floors, even in the blouses of passers-by. As to my ideals for the future, I dreamed of a life as an artist in abstract geometry, one who does not add anything to Creation but only makes its beauty manifest. I was convinced that great beauty lay hidden within the structures of perfect order. And I realized that in history of mankind there had been many predecessors who dedicated their lives to such an ideal. Already the craftsmen at the time of the great Pharaoh's realized great beauty in playing with the laws of symmetry, when decorating the tombs of the elite of their time. Absolute masters in this genre were the Islamic decoration artists during prime time of Islamic culture. Never reached geometric decoration art greater beaut. And in our days, the Dutch graphic artist Maurits Escher astonished the world with his masterpieces of regular plane division. Worldwide people are familiar with his slide-mirroring horses and animals with law cuddliness that hook into each other with their limbs.

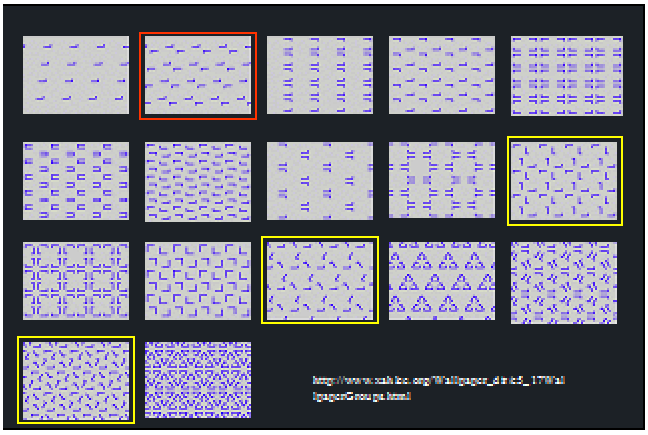


Figure 1*: The seventeen*

All these artists had been playing with the 'regimes' that rule in the different plane symmetries. With Maurits Escher I share that deep feeling of loneliness which seems indissolubly linked with a fascination for regularity and order. Escher once put this feeling into words in his beautiful, expressive, somewhat megalomaniac way:

> **"All alone, I walk in this magnificent garden which does not belong to me. Though its gate is wide open to everyone, I sojourn there in refreshing, but at the same time dejecting loneliness. I have, therefore, testified to the existence of this Garden of Eden for many years..., without expecting to attract many visitors. For the things by which I am truly fascinated and which I experience as beautiful are obviously considered dull and dry by others."**

In past decades I was a frequent visitor of that garden. But I was going to love there quite a different kind of beauty than Escher. A beauty that was linked to a wondrous kind of binary sequences which were inherent to all 17 plane symmetries. I call them Christoffel sequences, after the man who as first publishes about them.

My eye was at first struck by the beauty and elegance of these sequences in the summer of 1969. I was just out of military service and should have felt free then. But instead my obsession with symmetrical ordering took me in its grasp more tightly than ever before. It ruled like a tyrant over everything I thought, felt, and wished for. So it happened that, one day, when I was frantically trying to grasp the essence of a pattern of Greek crosses designed in a grid of squares, I made a remarkable discovery. By drawing 'deflection' lines in it and conceiving a cross as a closed stripe path, the length ratio 1/2 appeared (Figure 2). Every change in that ratio resulted in another stripe path (Figure 3).

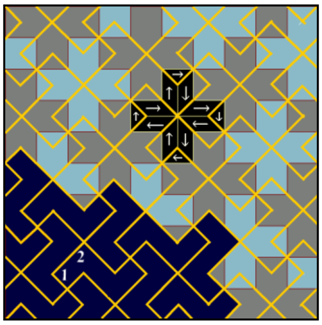


Figure 2: By drawing 'deflection' lines in it and conceiving a cross as a closed stripe path, the length ratio 1/2 appeared

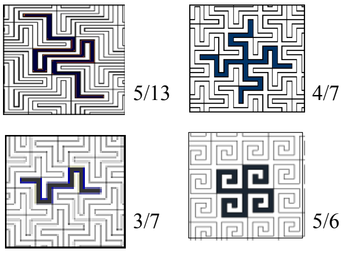


Figure 3: Every ratio results in a different stripe path.

All stripe paths had the character of a concave polygon, besides the most basic, which was the square. At the first level above the basic one, the stripe paths could be conceived as stages in spiral winding around that square - the cross, the zigzag, the swastika, etc. - reminding of the meanders on Greek pottery from antiquity. At higher levels (when ratios became more complex), the shapes of these first level stripe paths, being stages in simple spiral winding around the square, became themselves 'molds' for spiral winding at a still higher level. And so on. The different levels appeared to be related to the stages in the continued fraction of the respective fractions. This kind of stripe paths also could be generated in a grid of triangles. Especially there, the resulting paths could have an amazing beauty and elegance (Figure 4).

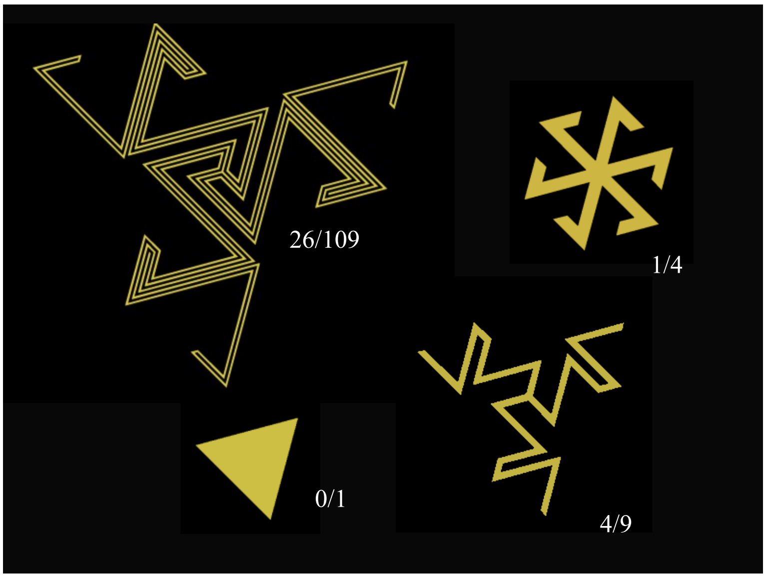


***Figure 4**: Especially in p6, beautiful convex polygons arise.*

Nearly half a century now I'm under the spell of this phenomenon. It took me years to describe the structure of these beautiful concave polygons with a mathematical simplicity and transparency that corresponds with the elegance and beauty that they 'radiate'. I guess the job could have been done in shorter time if I had a mathematical shaped mind. But my aptitude for mathematics is poor. I always got bad grades for it in high school. Anyway, during that long journey I often had to think of the warning of the Hungarian mathematician Farkas Bolyai to his son Janos against trying to prove the Euclidean axiom that there can be only one parallel to a line through a point outside of it:

. . You should not tempt the parallels in this way, I know this way until its end—I also have measured this bottomless night. 1 have lost in it every light, every joy of my life—... You should shy away from it as if from lewd intercourse, it can deprive you of all your leisure, your health, your peace of mind and your entire happiness.— This infinite darkness might perhaps absorb a thousand giant Newtonian towers, it will never be light on earth, and the miserable human race will never have something absolutely pure, not even geometry... [Stáckel, pp. 76–77].

Ultimately my night appeared not as bottomless as that which Farkas Bolyai foresaw for his son. The light that I ultimately was able to kindle came from a number of quasi mathematical concepts which I developed all by myself. The most bizarre thing that happened during my long search was that I all by myself invented the principle of continued fraction, having no idea that this arithmetic tool was known to clockmakers already since several centuries. Christiaan Huygens had already used this tool in 1682, in the design of his planetarium.

alternation of S-events and L-events

My stripe paths in their meandering though the plane, alternately traverse L(onger) and S(horter deflection lines. They can be conceived as generating a binary sequence of S- an L-events. In the example in Figure 5 the ratio between the longer and shorter deflection lines is 5/17 and the resulting sequence is LLLLSLLLSLLLLSLLLSLLLS. It's immediately clear that the spreading of the 17 L-events over the 5 S-events is as evenly as possible.

Many years I racked my brain about the precise structure of these sequences in relation to that ratio. In mathematics lessons at high school we never heard about these wonderful sequences.

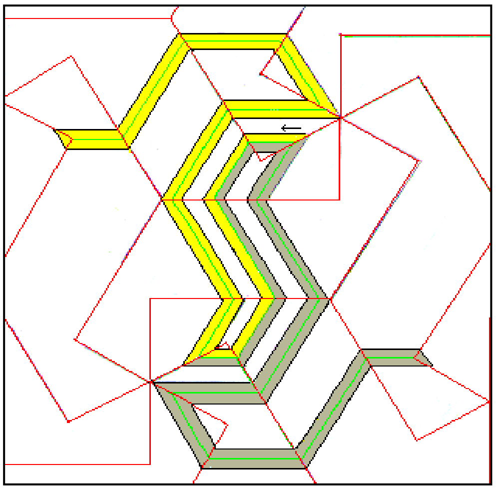


***Figure 5** :Stripe path generating a sequence of S- and L-events*

Christoffel sequences in daily life

In daily life we are more often confronted with them than we might think because of this unfamiliarity. One of the most impressive things in Creation, the interaction of the time schedule of sun and moon, has the same characteristics as the sequences generated by my stripe paths (Figure 6).

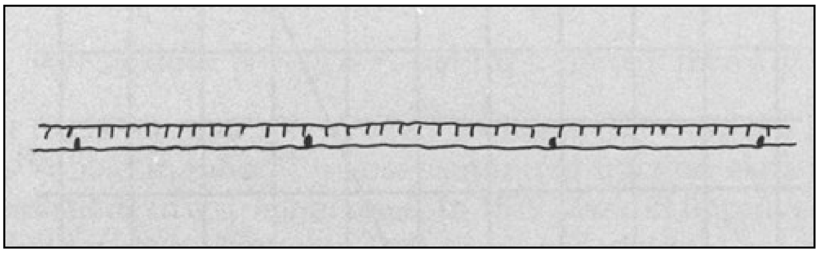


***Figure 6**: Tally marks showing lunar months and solar years. Source* {1}

But we encounter them also in less majestic events. When you draw a straight line at the tiled wall in your kitchen or bathroom, you also have generated such a sequence (Figure 7 [1]).

[1] Cartoon Bathroom Interiorvirtualhorizonstudio

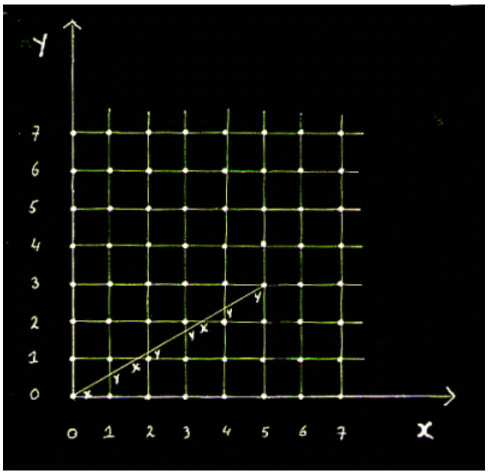


***Figure** 7: Christoffel sequences in the bathroom.*

And also the pedestrians who daily cross the public squares of our great cities generate Christoffel sequences (Figure 8). Especially when a plane symmetry is inherent to the pavement pattern of those squares. We explain that in the next section.

***Figure 8:** Also the pedestrians who daily cross the public squares of our great cities generate Christoffel sequences.*

Elwin Christoffel

Past year I heard that these sequences are already known in mathematics since 1875. Caroline Series showed me the way to the mathematical conceptualizations regarding this phenomenon, in a fascinating article[2] in this journal that was about the 'geometry of the Markov numbers' {1}. And it was she who pointed me to Elwin Christoffel as the first who wrote about these sequences {2}.

[2] Thanks to Dr C. Kraaikamp of the Technical University Delft (NL) who made me aware of this article.

2 Christoffel sequences

2.1 Christoffel's observations

Elwin Bruno Christoffel

So it happened that last year Elwin Bruno Christoffel [3] posthumously came in my life. Judging from what I read about him, he certainly was not a sweetie for his social environment. His biographers[4], marked him as “a lonely man, ... shy, distrustful, unsociable, irritable and brusque". In characterological terms there was little reason to identify with this man. But also in terms of content, there certainly did not open a whole new world to me. The algorithm he used to generate these sequences factually masked their intrinsic beauty and universal character. And his observations about their internal structure remained very global. It were Henry Smith one year later, and Caroline Series roughly a century later, who, rather independently of each other, hit the essence of that structure. Still there are some very useful elements in Christoffel's article, so let me shortly sketch the content.
To generate his sequences, Christoffel used an algorithm based on modular arithmetic. It's a calculation method in which one counts with positive integers which have an upper limit. That upper limit is called the 'modulus'. If the outcome of an arithmetic operation is equal to or greater than the modulus, this is subtracted one of more times from that outcome, until a number smaller than the modulus is obtained again. For example, when the modulus is 5 than 3 x 4 = 2. The way a clock measures time is the most familiar example of this way of numbering.
Working with this principle, Christoffel used the following algorithm. Given two relative prime integers *a* and *b* (both not divisible by a same number other than 1), the successive elements (r_n) in the sequence are: $r_1 = a \bmod b$, $r_2 = 2a \bmod b$, $r_3 = 3a \bmod b$, etc.. Before he analyzed these sequences, he converted them in two-letter-sequences by replacing an integer by the letter c (crescit) when it had a higher value than the preceding one and by a letter d (decrescit) when it had a lower value[5]. For example when a = 4 and b = 11:

4	8	1	5	9	2	6	10	3	7	0	[4...]
c	c	d	c	c	d	c	c	d	c	d	[c....]

It is directly clear that in one period of such a sequence the number of d's equals *a* and the number of c's equals *b-a*.
Christoffel offers only a few global notions about the structure of his sequences. He emphasized their periodicity and pointed to the standard presence of a main part ('pars principalis') in a period, which he called 'symmetrical'. To get this main part one needed to disregard the first and the last letter in a period, always being *c* and *d* respectively. In the example above:

c cdcc |d| ccdc d

In the mid lies the point of reversion (rp). That point coincides with the letter c, with the letter d or with 'no letter' . We can consider the letters c, d and 'no letter' as possible reversion-point-particles (rpp). Christoffel introduced a notation system for these. When a letter c or d coincides with the rp, these rpp's are notated as |c| and |d|. When 'no letter' coincides with the rp, this rpp is notated as |.
Whether an rp coincides with a letter c, a letter d or with 'no letter' , depends on the a/(b-a) ratio.
What Christoffel in his time called a 'symmetrical main part', nowadays is called a palindrome.
Mathematicians after Christoffel, especially in the area of ' Combinatorics on Words', paid a lot of

[3] Photo: www.eifelzeiting.de
[4] P L Butzer, An outline of the life and work of E B Christoffel, in P L Butzer and F Fehér (eds.), *E B Christoffel: The influence of his work on mathematics and the physical sciences* (Basel- Boston- Stuttgart, 1981), 2-29.
[5] Factually his algorithm was a little more complex: you should substitute a number by the letter d when the next number decreases in value, not the current one. I never have understood why he made it unnecessary complex.

attention to this palindrome in Christoffel sequences. But they didn't take over Christoffel's suggestion to treat the rpp as a separate element within it. Until today they cling on the popular but scientifically to simple idea that a palindrome is a word that has the same sequence of letters whether you read from front to back or vice versa. The rpp's in Christoffel sequences are one of the most intriguing phenomenon of discrete mathematics. I hope the reader will agree with me after reading this paper.

Christoffels' observation that there standard was a palindrome present in the main part of his sequences and that one could single out r.p.p's in that palindrome, inspired me a lot. It opened the road to a far deeper understanding of my spirals and crystals. And while making my way on that road, I discovered that also another type of palindrome was inherent to Christoffe's sequences, but only when these sequences are two-sided infinite, which is always the case in plane symmetries . This second type appeared the ultimate key to understanding. That Christoffel did not notice it, is most likely due to the fact that he used an algorithm that led to one-sided infinite sequences.

2.2 Broadening of Christoffel's original idea.
more transparency in the algorithm

There is a lot of mathematical elegance in Christoffel sequences. The chilly technical jargon in which Christoffel talks about them (in the best tradition of mathematics), does not pave the way for much esthetic appreciation. Before I became acquainted with Christoffels algorithm, I had developed my own. In my unsophisticated mind it came out on a jumping process on a circle. You simply place the S-events adjacent to each other on that circle (forming an 'S-string') and so the L-events (forming an 'L-string'). Figure 9 shows this for the example s/$\boldsymbol{l}$ = 3/7 , whereby *s* indicates the number of S-events and *l* indicates the number of L-events. The only thing you need to do is to hop clockwise or counter clockwise over the circle from letter position to letter position, whereby after each hop you come down at a place that is *s* letter positions away from the place from which you started that hop. The successive letter positions on which you come down after a hop, form a Christoffel sequence. Thereby it doesn't matter at which letter position on the circle you start and whether you hop in clockwisedirection or in counterclockwise direction. In the example the period of the generated sequence is (starting from the arrow) LLLSLLSLLS. After so many years of using this algorithm, it every time anew feels as a miracle that in such a way generated sequences offers the best possible solution for an as evenly as possible spreading of L-events over S-events. These solutions have some intrinsic beauty, which can be made visible in geometric shapes. For example in the spreading of obtuse angles over sharp angles in the concave polygons in Figure 4. Each of these is a masterpiece in ‘striving’ for balance.

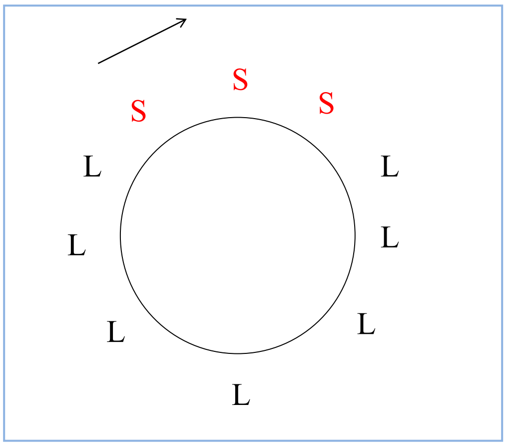


***Figure 9**: Simplification of Christoffel's algorithm. Two letters lie on the mirror axis through the circle .*

two-sided-infinite
Christoffel wrote about his sequences as if he saw them as one-sided-infinite. May be this was because of the use of the principle of modular arithmetic in his algorithm. But they can as well be two-sided-infinite. It depends on which area of reality you encounter them. In Escher's garden (= the seventeen plane symmetries) they by definition show up in a two-sided-infinite appearance,

↓ ↓ ↓ ↓

infinity ← ...... L L L S L L S L L S L L L S L L S L L ...... → infinity

***Figure 10:** Two-sided-infinite Christoffel sequence for the example* s/l = 3/7. *De red and green arrows indicate the letter positions from which you encounter the same ordering of letters, whether you go to the infinity in left or in right direction.*

with no fixed starting point for a period (Figure 10). When you walk through such a sequence, starting from a letter indicated by a red or green arrow, it doesn't matter if you are walking in the one or in the other direction into infinity. You encounter the same ordering in the successive letters. These letters lie on the mirror axis in a circular presentation of the period (Figure 11).

sequentiality and periodicity
Two sided infinity implies that there is no 'natural' starting point for a period in the Christoffel sequences. To make sense out of this lack of 'structural hold' in concepts like 'beginning' or 'ending', we introduce the concept 'sequentiality'. It is, so to speak, the house in which all the possible versions of the period have shelter. The reader shouldn't worry about the nebulosity of this. I 'm going to make it concrete!

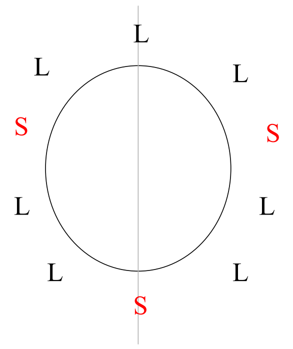


Figure 11: *Sequentiality of the example 3/7.*

The period of a two-sided infinite sequence can be presented on a circle, as just stated. That means: we placeon that circle a piece of s + ***l*** adjacent letters, which is cut out from that the sequence at a random place. Where in the sequence you choose the beginning of that piece, does not matter. We shall call these s + ***l*** letters, presented on a circle, without conceiving a beginning and ending, the generic sequentiality of the sequence. Figure 11 shows this sequentiality for the example s/***l*** =3/7.

1)	SLLLSLLSLL
2)	LSLLLSLLSL
3)	LLSLLLSLLS
4)	SLLSLLLSLL
5)	LSLLSLLLSL
6)	LLSLLSLLLS
7)	SLLSLLSLLL
8)	LSLLSLLSLL
9)	LLSLLSLLSL
10)	LLLSLLSLLS

Table I*: There are s +* ***l*** *different variants of the period within the sequentiality of a Christoffel sequence. This table shows the variants within the sequentiality of the example 3/7. In variants 7 and 10 an i-Pal is locked up. The variants 3 ,4,8 and 9 coincide with the o-Pal.*

Within this sequentiality you can freely decide on which letter position the period starts. There are *s + l* different letter positions that you can take as starting point. So there are *s + l* different variants of the period. Because there is always a mirror axis in the ordering of the letters in the sequentiality, it doesn't matter whether you go clockwise or contra clockwise through the sequentiality. In both

directions there are the same s + l variants. Table I shows all the variants for s/l = 3/7. They are pair wise each other's reversal: 1 and 6 , 2 and 5, 3 and 4, 7 and 10, 8 and 9.
May be you, reader, are not amused by this bone-dry, phone book-like substance. But don't worry , the same goes for me. But we should have some feeling for this type of differentiation within the sequentiality.

two types of palindromes
When we see Christoffel sequences as two-side-infinite, there shows up another palindrome beside the one Christoffel himself observed. Let's call the one observed by Christoffel the i-PAL because it is locked up in a period and thus inner-period-oriented. The new appearing palindrome too consists of two parts which are each other's reversal. Different from the i-PAL, it has two rp's. These lie always on the mirror axis in the circular presentation of the sequentiality (see Figure 11). It fully coincides with a period, whereby the pattern of reversal stretches in adjacent periods It's outer period oriented so let's call this one an o-PAL. Figure 12 shows the general model of the i-PAL and the o-PAL. In both palindromes each rp is 'seat' for one of three possible rpp's: a letter L, a letter S or 'no letter'[6]. Each of these three options occurs once in a Christoffel sequence. So in the example in Figure 12 a letter S and ' no letter' are the rpp's in the o-PAL and a letter L is the rpp in the i-PAL. Which two are involved in the o-PAL and which one in the i-PAL, depends on the 'odd' or 'even' of the numbers in the respective fraction.

Example : LLLSLLSLLS

i-PAL:

L | …. |rp| …. | S

L **LLSL** |L| **LSLL** S

o-PAL:

| |rp| ….. |rp| …. |

| | **LSLL** |S| **LLSL**

Figure 12. *: The two types of palindromes.*

Complemented Christoffel sequences.

In plane symmetries, Christoffel sequences often are composed of three letters instead of two. Beside the letters S and L, we use the letter M in these cases , which refers to *medium.* A quite elegant algorithm for the generation of this type of Christoffel sequences is a stripe path running

[6] Different from Christoffel's notation, we enclose all three types of rpp's in a pair of vertical stripes, also the type 'no letter' (see Figure).

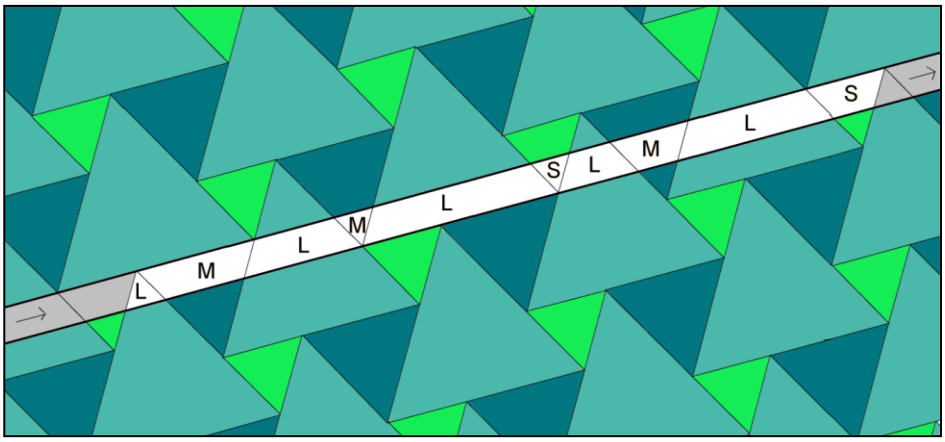


Figure 13*: Algorithm for the generation of three letter Christoffel sequences.*

through a tessellation composed of three triangles in three different seizes (Figure 13)[7]. The three letters refer to the three sizes that we know so well from our search in the racks of clothing shops. The letter L is fully dependent on the occurrence of the two others. It occurs before every letter M and letter S. By consequence, the resulting Christoffel sequence still can be considered as binary, with LS and LM as elements. The fraction *s/m/l* = 3/4/7 for example results in LM LM LS LM LS LM LS. And the fraction *s/m/l* = 2/3/5 in LM LM LS LM LS. We call these three letter sequences 'complemented Christoffel sequences' and the respective fractions 'complemented fractions'. A complemented fraction in general has three characteristics: 1) ℓ , *m* and s are relative prime; 2) $m = \ell$ - s; 3) one of the three value's is even and the other two are odd.
Figure14 shows the i-PAL (red mirror-axis) and the o-Pal (green mirror-axis) in this complemented version of the Christoffel sequence in the examples s/m/l = 3/4/7 and /m/l = 2/3/5. To get the i-PAL in these three-letter sequences, we need to remove three letters instead of two. In the examples these letters are colored gray.

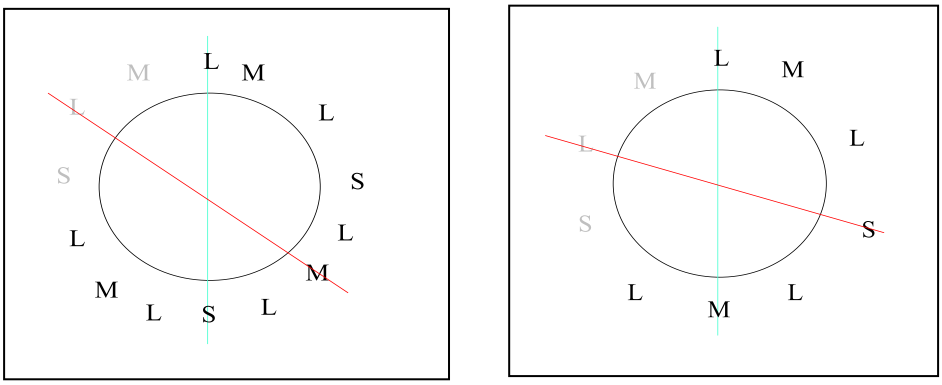


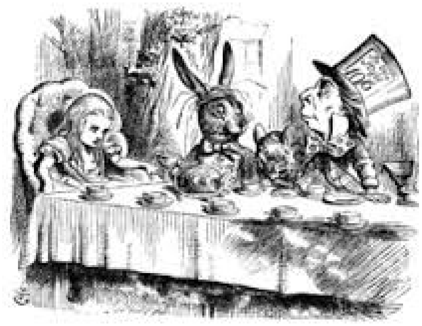
Figure 14: *The i-PAL and the o-PAL in the complemented version of the Christoffel sequences. The examples are 3/4/7 and 2/3/5.*

In the complemented version of the Christoffel sequence all three rp's coincide with a letter. In the example s/m/l = 3/4/7, the one in the i-PAL (red mirror axis) coincides with an M and the two in the o-PAL (green mirror axis) with an L and an S. In the example s/m/l = 2/3/5 the one in the i-PAL coincides with an S and the two in the o-PAL with an L and an M.
As discussed in the next subsection, to get grasp on the internal structure of Christoffel sequences the two binary letter elements must be considered as basic palindromes and thus as symmetrical in the ordering of letters. A notation in letter halves matches with this symmetry. LS is notated as $\frac{L}{2}\frac{S}{2}\ \frac{S}{2}\frac{L}{2}$

[7] L, M and S are integers, specifying the number of length units in the sides of the respective triangles.

and LM as $\frac{L}{2}\frac{M}{2}\ \frac{M}{2}\frac{L}{2}$ [8]. These basic palindromes are 'empty', which means that they only consists of the four rp-halves which all are seat of a letter half. As we further shall make clear in the next section, Christoffel sequences can best be conceived as built up of palindrome <u>halves</u>. The most basic palindrome halves are $\frac{S}{2}\frac{L}{2}$ and $\frac{M}{2}\frac{L}{2}$.

2.3 Fathoming the intrinsic structure.



In subsection 2.2 some global characteristics of Christoffel sequences were discussed: their infinity, the main structure of their sequentiality and the types of palindromes which occur within them. But above all Christoffel sequences are intriguing because of the complexity of their internal structure. Despite the very transparent algorithm for the generation of these sequences, as presented in Figure 9, their internal structure is shrouded in an air of mystery. It took me years to fathom it. During that long journey I often felt like Alice in Wonderland, wandering through a weird world full of odd phenomena. A world in which pairs of opposites like *less* and *more*, *odd* and *even*, *left* and *right*, *clockwise* and *counterclockwise*, *upward* and *downward majority* and *minority*, *inward* and *outward*, *entrance* and *exit*, *beginning* and *ending,* all are very crucial in the understanding the complexity of Christoffel sequences and, as a derivate, the complexity of the geometrical shapes that can be brought forward as a geometrical expression of that internal structure. Where you can look at things as being something static or as being a flow. Where things are asymmetric and at the same time look symmetric. Where fractions are built up in layers. Where the difference between ratios and fractions become quite a philosophical issue. Where polygons are the keystones of coordinate systems.

2.3.1 Layeredness in the alternation of n and 1.

When I started my journey, it became soon clear that the stripe paths which I had discovered in principle had a layered structure. And I began to realize that this layeredness could be related to the fraction s/ℓ [9] after decomposing it by means of a process of continued division[10]. The decomposition schedule, which in mathematics is called a continued fraction or Kettenbrüche, is presented in Figure 15. The series of partial quotients (the successive values for *a*) represents the fraction and usually is written as [*a1, a2, a3,* ..]. Every premature termination of the process of repeated division results in a fraction that is a rough approximation of the final fraction. These are the so called 'convergents' of a continued fraction: the 1st convergent is [a1], the 2nd is [a1, a2], etc... .

general model:	application to 5/17:
$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$	$\cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2}}}$

Figure 15*: Model continued fraction.*

8 The Christoffel sequence *s/l/m* = 2/3/5 for example i= in this idiom becomes: $\frac{L}{2}\frac{M}{2}\ \frac{M}{2}\frac{L}{2}\quad \frac{L}{2}\frac{S}{2}\ \frac{S}{2}\frac{L}{2}\quad \frac{L}{2}\frac{M}{2}\ \frac{M}{2}\frac{L}{2}\quad \frac{L}{2}\frac{S}{2}\ \frac{S}{2}\frac{L}{2}\quad \frac{L}{2}\frac{M}{2}\ \frac{M}{2}\frac{L}{2}$. Of course this is a very laborious way to present a Christoffel sequence, so we can choose a more aggregated way: $\frac{L}{2}$ MLSLMLSLM$\frac{L}{2}$.
It does justice to the symmetry in the letter ordering of the sequence as a whole but not in the letter ordering per binary letter element. The following notation satisfies more, although it is again rather laborious: $\frac{L}{2}\mathrm{M}\frac{L}{2}\quad \frac{L}{2}\mathrm{S}\frac{L}{2}\quad \frac{L}{2}\mathrm{M}\frac{L}{2}\quad \frac{L}{2}\mathrm{S}\frac{L}{2}\quad \frac{L}{2}\mathrm{M}\frac{L}{2}$ .But whatever notation we choose, we in must realize that the most laborious shows the essence of complemented Christoffel sequences.

9 In the first beginning I only worked with two-letter-sequences.

10 Dividing denominator by numerator, numerator by remainder, etc. until the remainder reached zero. Each division delivers a partial quotient.

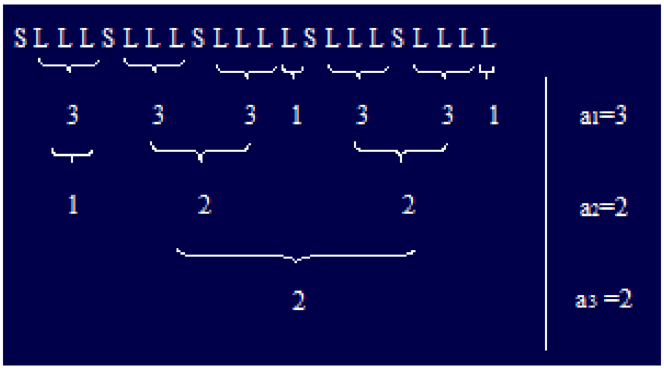


Figure 16 *: Layered character*

What had to be done finally was to describe the layered structure in such way that it could be related quite transparently to the series of partial quotients. That looked like a piece of cake but it took me years to get that far. The problem kept me bothering like a smouldering piece of mischief which could, at certain times, turn into a roaring blaze. During a holiday, while on a trip to Hungary (obsessions do not slumber in holidays), on a dingy little road along the river Donau, at the Austrian-Hungarian border, I found the solution (Figure 16).

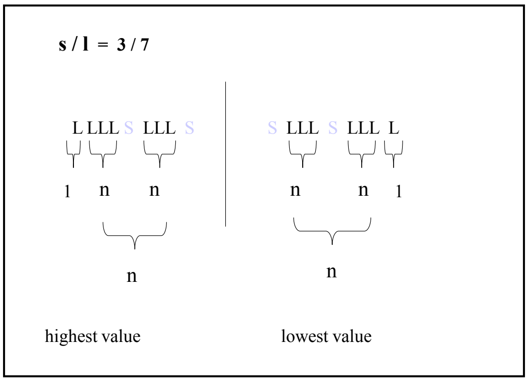


Figure 17: *The diagram can be based on two variants in the period-bound sequentiality. These two have respectively the lowest and highest value in lexicographic order. The two diagrams are each other's mirror image.*

In a Christoffel sequence between every two consecutive S-elements a string of L-elements (= L-string) shows up. The L-strings alternately contain n or n + 1 elements. When they contain n + 1 elements, we note n and 1 as separate numbers. At the following levels n and 1 themselves become the elements in the sequence. Now, 'n-strings' show up between every two consecutive '1-elements' , whereby these n-strings alternately contain n of n+1 elements. At the last level the number of 'n-elements' between two '1-elements' is constant because at this level there is but one pair[11] of consecutive '1-elements' . The relation between the continued fraction and this rewritten L/S-series now became immediately clear: the number of stages in the continued fraction equals the number of layers in the alternation of n- and 1-elements and the partial quotient at each stage in the continued fraction is equal to the value for n at the regarding layer (Figure 17).

Factually there are two variants of the decomposition diagram in terms of the alternation of n and 1. The one is based on the variant of the period which has the lowest lexicographic value and the other is based on the variant with the highest lexicographic value. They are each other's mirror image.

The layered decomposition of a Christoffel sequences in terms of alternation of n and 1 can be converted [12] in a decomposition in which periods of lower order Christoffel sequences are the building blocks[13]. Thereby also the single elements L and S are considered as periods of a Christoffel sequence, while strictly they are not. Figure 18 shows the example *s/l* = 5/17. At every layer i in the decomposition diagram, two different periods show up: one is in majority and one is in minority. Let's call them 'majors' and ' minors' respectively. The majors at a certain level return as minors at the next level. Figure 19 presents the *s/l* ratios of the majors and minors at the different layers and the series of partial quotients that represent these ratio's. The first position within these series remains empty because it refers to the majors at layer 0, which have no partial quotient. As Figure 19 makes clear, at each layer in the decomposition diagram the *s/l*-ratio of the majors has one level more in its continued fraction than the S/L-ratio of the minors.

---

[11] The last part of this pair of consecutive S-symbols in this case belongs to the next period. So in a circular presentation there is but one '1' -symbol followed by n symbols n-symbol and one S-symbol and n L-symbols.

[12] Therefore, at the first layer, we need to add one S-event to the string of L-events that we have called the n-element. It is the S-event that lies direct adjacent[12] to that string. Then , at each layer, inclusive that first, an 'n'- element' becomes a period of a Christoffel sequence with i layers and an '1'- element becomes a period of a Christoffel sequence with i-1 layers.

[13] At the basic level, that we call level 0, the letters L are the majors and the letters S the minors. L is indicated as P_0 because it is the major at layer 0. S has no layer where it is the major, so we indicate it simply as P. At some places in the paper we use the symbol P when talking about periods in general (P's), not referring specificly to the letter S as a period. The context in these case makes clear that we are talking in general.

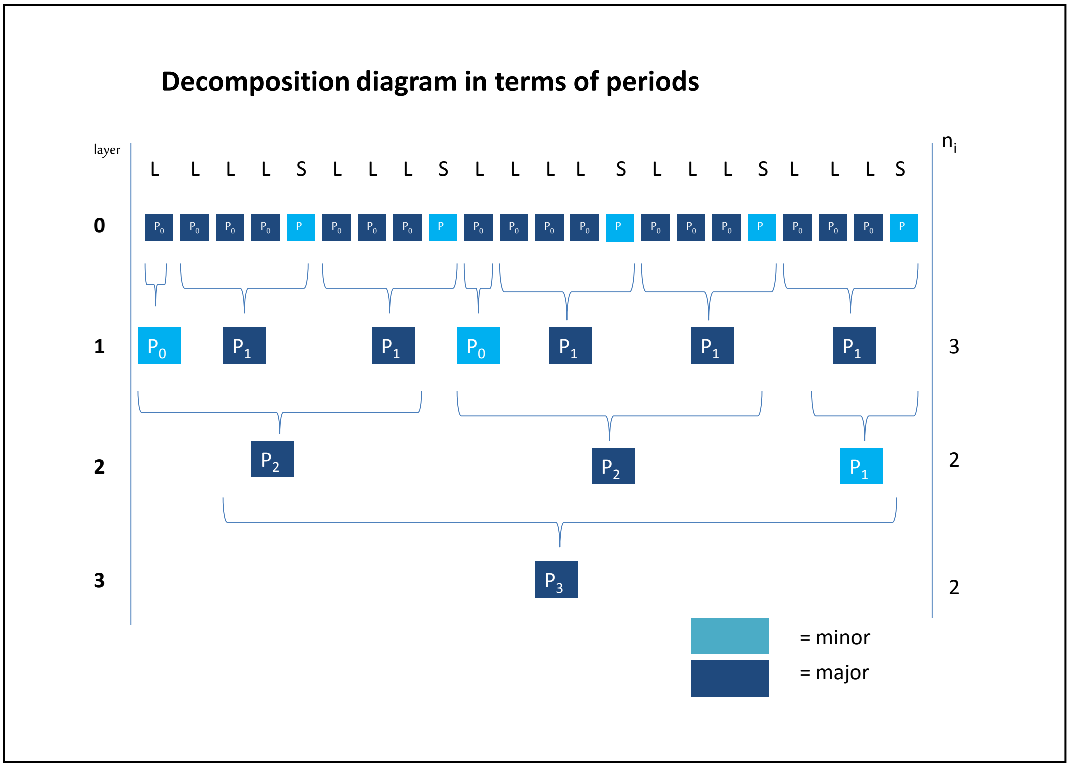


Figure 18: Decomposition diagram in terms of periods, for the ratio s/ℓ = 5/17.

Explicitation of the *s/l*- ratios of the periods

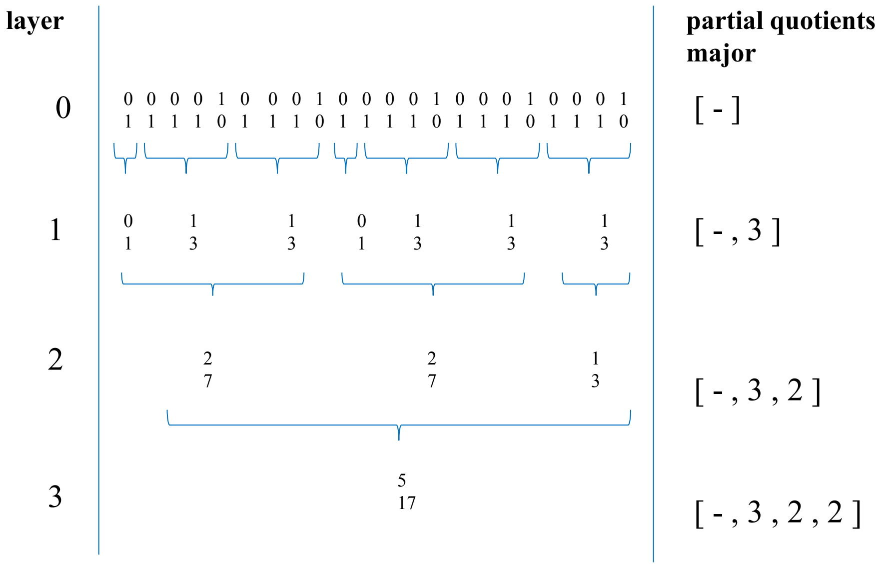


Figure 19: *Explicitation of the s/l- ratios of the periods in the diagram of Figure 18.*

The accumulates in Figure 18 show that at every arbitrary layer i , each of its majors is composed of n_i majors from layer i-1 , supplemented by one minor from that layer.
The partial quotients of the major at a certain level contain all the information about the components of which it is composed. Let's illustrate this with the final major in the diagram in Figure 18. As indicated in Figure19, the series of partial quotients that belongs to this major is [- , 3, 2, 2]. The last partial quotient indicates the number of majors at level i-1 that we see within the accumulate. So in this example there are two of them. When we remove the last partial quotient, thus [- , 3 , 2 , $\cancel{2}$] , the remaining series specifies the *s/l*-ratio of these majors. When we also remove the second to last partial quotient, thus [- , 3 , $\cancel{2}$, $\cancel{2}$] the remaining series specifies the *s/l*-ratio of the minor. In this example it is 1/3.

when value(s) 1 show up in the series of partial quotients.
In the example in Figure 18 and 19, the *s/l* -ratio of the majors has one level more than that of the minors, in terms of stages in the continued fraction (= number of partial quotients). But that's not always the rule, especially when value's 1 show up in the tail of a continued fraction.When the last partial quotient in a series of partial quotients is 1, you can add it to the second to last partial quotient, without changing the values in the fraction. For example [2,1,1] represents 2/5 but [2,2] also represents 2/5. In the same way, when the second to last partial quotient is 1, you can add it to the third to last partial quotients, when you have removed the last partial quotients in order to determine the S/L ratio of the majors within the accumulate. And a partial quotient with value 1 in the third to last position can be added to the partial quotient in the fourth to last position, when you have removed the last two values in the series of partial quotients in order to determine the *s/l*-ratio of the minor within the accumulate. Table II shows the possible consequences. Minor and majors within that accumulate can differ 0, 1 or 2 levels from each other, dependent on the presence or absence of a value 1 in the second to last and third to last position in the series of partial quotients.

major_i	**components within accumulate**		**difference in number of levels of majors and minor at level i-1**
	major_{i-1}	**minor_{i-1}**	
[1,1,2,1,2]	[1,1,3]	[1,1,2]	0
[1,2,1,1,2]	[1,2,2]	[1,3]	1
[1,1,1,2,2]	[1,1,1,2]	[1,2]	2

Table II: *Impact of the showing up of a value 1 as second to last and/or third to last partial quotient in the series of partial major quotients that represents a major, on the components .*

All fractions are part of a twin
In the accumulates of any arbitrary decomposition diagram like that in Figure 18, we see a concatenation of majors, which is supplemented by one minor. As stated, this composition is specified by the series of partial quotients of the major to which the accumulate belongs. That series always is part of a twin of which the other part represents the same concatenation of majors, but a different minor. In one of the two series a value 1 is split of the second to last partial quotient. Let's illustrate this with the example 2/7 of which the twin partner is 3/8. The series of partial quotients of 2/7 is [- , 3 , 2]. When we split of a value 1 of the second to last quotient, we get [- , 2 , 1 , 2] , which are the quotients of 3/8. When we remove the last partial quotient in both series, the remaining series of partial quotients in both cases is [- , 3], which means that *s/l*-ratio in both cases is 1/3. So they represent the same concatenation of majors. But when we remove also the second to last partial quotient, in order to find the *s/l*-ratio of the minor, the remaining series are different. In case of 2/7 it is

it is [-] and in the case of 3/8 it is [- , 2] . The *s/l*-ratio of the minor of 2/7 is 0/1 and that of 3/8 is 1/2. In the case of 2/7 minor and major differ 1 level; in the case of 3/8 minor and major have the same level. Only the value of the repsective partial quotient differs 1, in this last case.

positioning of Christoffel sequences within the Carthesion Coordinate system

Christoffel sequences can in a very elegant way be ordered in a Carthesian coordinate system (Figure 20). The horizontal coordinate is called the L(arger) -coordinate; the vertical one the S(maller) coordinate. It means that values on the S coordinate are always smaller than those on the L-coordinate, except for the points lying on the diagonal. All Christoffel sequences can be represented by a pair of integers that are positioned on the coordinates. The integers are relative prime[14]. In the coordinate system these are the blue points. The red points represents the remaining pairs of integers which are not relative prime.

From the origin (0,0) we can draw an infinite line through any arbitrary blue point. After this line has passed this blue point, it goes through an infinite number of red points. The coordinates of these are a multiple of the coordinates of the blue point. For example when we draw an infinite line from (0,0) through the blue point (1,2), after traversing this point it goes successively through the red points with coordinate pairs (2,4) , (3,6) , (4,8) , etc.. By definition the red points do not represent Christoffel sequences. Each represent the concatenation of majors. Such a concatenation of majors

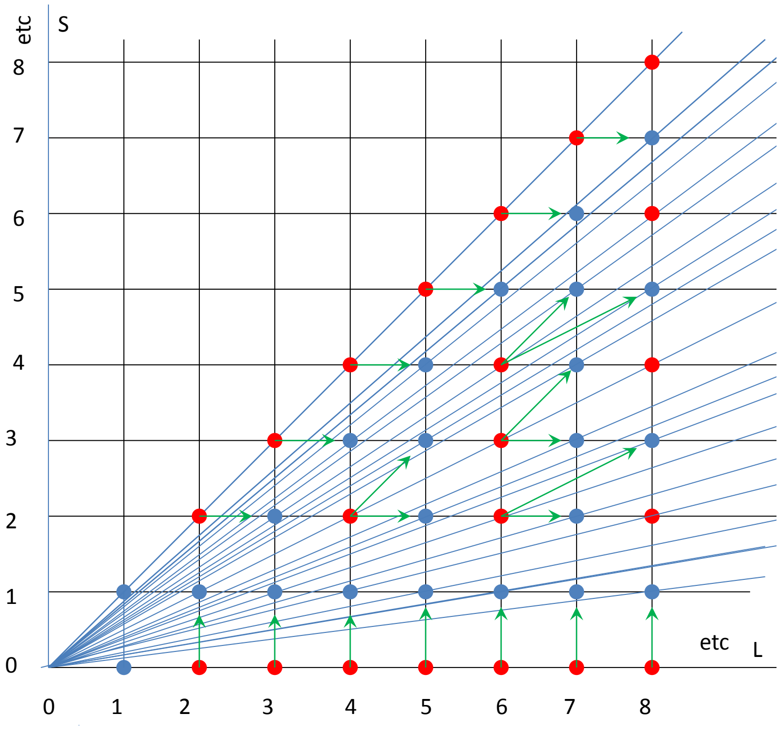


Figure 20: *Positioning of all Christoffel sequences within a Carthesian Coordinate System, making manifest the last layer in their built up.*

makes up the main part of every Christoffel sequence. By adding of a minor to it, a new bleu point is reached, representing a new Christoffel sequence. Reasoning form a certain red point, the blue line running from this point to the origin, represents the concatenation of majors and the two green lines running from this point to two different blue points, represent the two possible minors with which that

[14] The format in textual reference is (s,*l*).

concatenation of majors can be complemented into a new Christoffel sequence. We call these two the minor $^-$ and a minor $^+$. They are each other's complement in the major represented by the blue line. The two Christoffel sequences in question form a twin : they have the same concatenation of majors but differ in their minor.
Let's illustrate this with the example of the red point that has coordinates (3,6). The blue line to the origin represents a concatenation of three majors. Each of these majors on his own has the coordinate points (1,2). The minor $^-$ and the minor $^+$ which can be added to complement that concatenation into a new Christoffel sequence, on their own have coordinate points (0,1) and (1,1) respectively. They can be added not only to (3,6) but to every multitude of (1,2) in order to arrive at a new blue point. Adding them to (3,6), the fractions (3,7) and (4,7) show up respectively. The series of partial quotients of these fractions are [2,3] and [1,1,3] respectively. The major is represented by [2] and the two minors are represented by [-][15] and [1] respectively.
This build up of Christoffel sequences in terms of coordinate points applies to all Christoffel sequence. But for those of which the red point lies on the coordinate or on the diagonal between the coordinates, there is but one option as to the green lines: the minor $^-$. You cannot spit off a value 1 of the series of partial quotients in these cases.

2.3.2 Layered build-up of the palindromes.

Many years I was working with this concept of layeredness. It helped me to understand the structure of my stripe paths: the number of layers in spiral winding and the degree of spiral winding at each layer. But the dawn after the 'bottomless night' Farkas Bolyai feared for his son, still had not arrived. Although there shimmered some light on the horizon. I still was not able to understand the wonderful shapes of the concave polygons that came out of these layered structures, like those in Figure 4. Last year daylight finally broke through. I had taken note of Christoffel 's ideas and especially his observations that always a palindrome was present in the sequences he generated. I found out that this palindrome always showed up in the inner border of my stripe paths (see section 4, Figure 49) and that in their outer border another type of palindrome showed up. In section 2.2 (Figure 12) these two types of palindromes were introduced shortly as respectively the i-PAL and the o-PAL.

the o-PAL a 'jewel' in discrete mathematics
The o-PAL is a jewel in discrete mathematics. The model in Figure 12 only specifies its characteristics 'on the surface'. But o-PAL's have a complex structure and in essence can be seen as built up of lower order palindromes[16]. The decomposition diagram we introduced above (Figure 18) , with 5/17 as example, already shed some light on this layered built up, but it's asymmetrical in its appearance.
At the basic layer, we must work with the binary elements introduces before, which are already symmetrical ordered in themselves: $\frac{L}{2}S\frac{L}{2}$ and $\frac{L}{2}M\frac{L}{2}$. But we need to do more. The decomposition diagram as a whole must be transformed in a symmetrical one, with a mirror axis in its center. Let's do that for the example in Figure 18. Figure 21 presents the result. The diagram shows that not only at P_3 but also at the lower level periods P_2 and P_1 the respective letter strings have the shape of o-PAL's[17].
A crucial level in the understanding of the structure of the o-PAL is *the one but last layer*. In the example in Figure 21 that's layer 2. In that layer we see the highest level building blocks of which the final period P_4 is composed : P_2 P_1 P_2. In this layer there are always n P_{i-1} 's (the majors) that are symmetrical arranged around one P_{i-2} (the minor) . In this example $n_3 = 2$, so there are two P_2 building blocks arranged around one P_1 building block. From this example we can generalize to all

[15] The minor of [2,3] is not represented by a partial quotient because there are but 2 partial quotients in the series while the minor is represented by the third last. The respective ratio is 1/0. So the coordinates of this minor are (1,0) .
[16] Like we do with periods higher and lower levels, we indicate the level of palindromes by a subscript: PAL_1, PAL_2, etc. .The PAL that only exists of the letter S has no subscript, like the P(eriod) that only consists of the letter S. See also footnote 13.
[17] With a certain restriction. Not all letters lying at the extremity of palindromes at a lower levels have the character of an r.p.p at that level. A letter L at one of the extremities of a P_1 for example, only has the position of an r.p.p at that level when it functions as connection point between that P_1 and another P_1. When it functions as connection point between that P_1 and a P_0 it functions as an r.p.p at level 0 , connecting two P_0's of which one is a building block within that that P1. But when this P_1 is considered on its own, as shown in the lower part of Figure 33, the respective r.p.p grows in its position and becomes an r.p.p at level II.

other cases: the composition of any o-PAL at a certain level i can be understood as a number of P $_{i-1}$'s that are symmetrical arranged around a P $_{i-2}$.

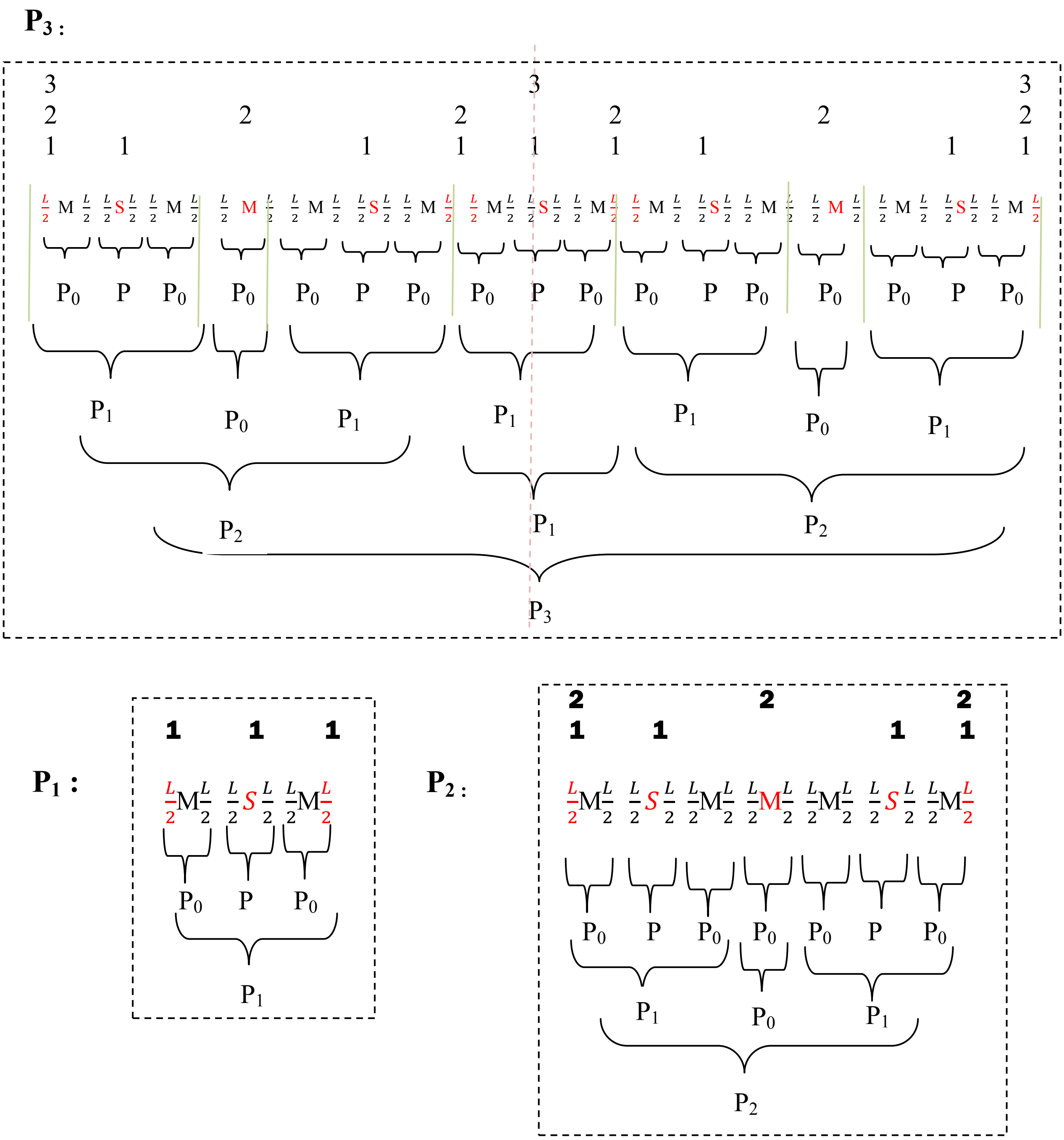


Figure 21: *Symmetrical decomposition diagram in terms of periods.*

general model
Figure 22 shows the general model for the built up of o-PAL's. The chosen example is n = 4 (n value for the final layer), but any arbitrary value for n can be chosen. Practice learns that it' s better to work with PAL-halves than with whole PAL's in the model. Let's denote these as $\frac{PAL}{2}$'s. The circular structure contains two final $\frac{PAL}{2}$'s , one left of the mirror axis and one right of it. We consider them as being of level i. They are both built up of n $\frac{PAL}{2}^{\,i-1}$'s which at the bottom of the circle are complemented by one $\frac{PAL}{2}^{\,i-2}$, which is colored red. The $\frac{PAL}{2}^{\,i-1}$ lying at the top of the circle, which is colored yellow, we call the *primary* one. It lies directly adjacent to the mirror axis. The remaining ones, which are colored gray, we call *supplementary*.

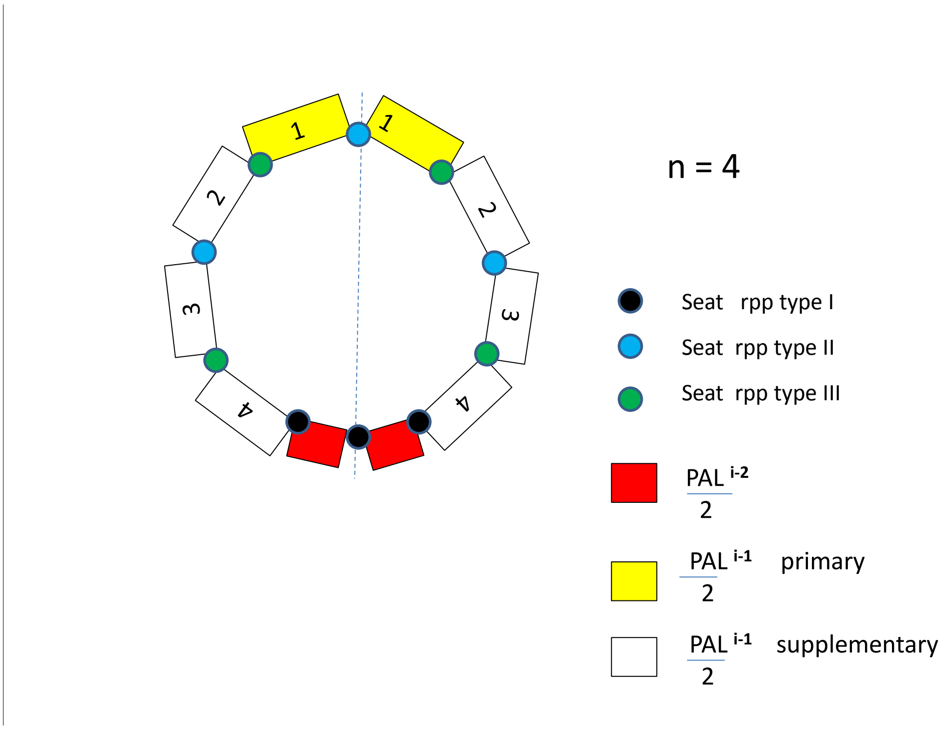


Figure 22: *General model for the internal structure of o-PAL's*

The connection points between the $\frac{PAL}{2}^{\,i-1}$'s are, by definition, the locations of the pair of rpp's at level i-1. At every other connection point this rpp is of the same type. Also the connection point between the two $\frac{PAL}{2}^{\,i-2}$'s is, by definition, the location for an rpp .This rpp is in all cases of another type then the two that have their location in the connection point between the PAL^{i-1} 's . The two connection points between a $\frac{PAL}{2}^{\,i-1}$ and a $\frac{PAL}{2}^{\,i-2}$ are r.p.p's at level i-2 (see footnote 15).
Two of the rpp's at level i-1 lie on the mirror axis. One is the rpp that has its location at the connection point between the two primary $\frac{PAL}{2}^{\,i-1}$'s . The other is the rpp that has its location at connection point between the two primary $\frac{PAL}{2}^{\,i-2}$'s. These two are also the rpp's of the PAL at level i which is the final palindrome. So they are 'active' at level i as well as at level i-1.
The model in Figure 22 not only applies to the built-up of the final $\frac{PAL}{2}$, but also to that of all lower level $\frac{PAL}{2}$'s that occur in the built up of that $\frac{PAL}{2}$. So we can color the decomposition diagram in Figure 21 in accordance with the model (Figure 23).

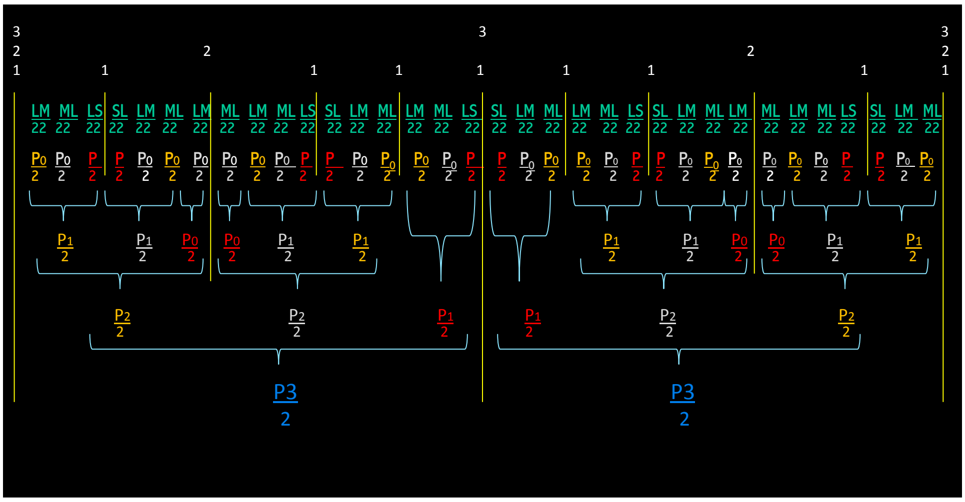


Figure 23: *The model not only applies to the final level but also to the lower levels, starting with level 2*

level 1 is lowest level

Level 1 is the lowest level to which the model is applicable. At that level the building blocks on the circle are the basic palindrome halves $\frac{L}{2}\ \frac{M}{2}$ $(\frac{P0}{2})$ and $\frac{L}{2}\frac{S}{2}$ $(\frac{P}{2})$. The lowest value n at this level, the value 1, represents the cross (Figure 24) which is one of the most emotional charged symbols in the cultural heritage of mankind. In section 4 more about the cross as geometric expression of the most simple palindrome

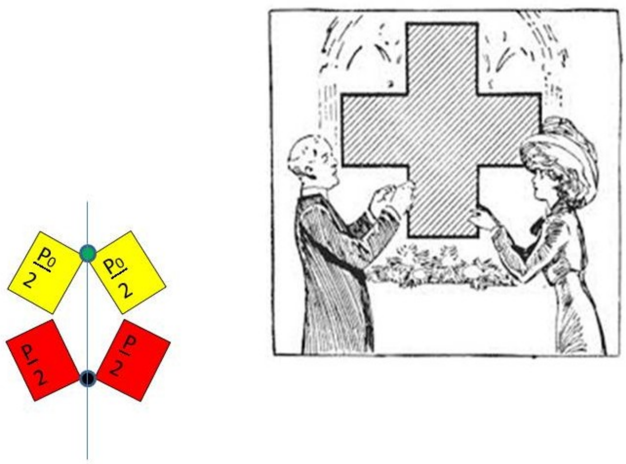


Figure 24: *The cross is the simplest not empty palindrome*

the i-PAL

Now we have grasp on the structure of the o-PAL's, it doesn't need much extra effort to get that grasp on the structure of the i-Pals. Only the built up of the final layer is different. For the lower levels the just described model for o-PAL's suffies. In its final level the i-PAL has but one rpp, which lies in the center. It is the connection point between two series of n_i $\frac{PALi-1}{2}$'s , one lying left and one right of the mirror axis, both ending in a P(rematurely) A(borted) P(eriod) at the same level (Figure 32).

$$PAP_{i-1} + \frac{PALi-1}{2}^{\,ni} \;\ldots\ldots\; \frac{PALi-1}{2}^{\,2} + \frac{PALi-1}{2}^{\,1} \;\Big\vdots\; \frac{PALi-1}{2}^{\,1} + \frac{PALi-1}{2}^{\,2} \;\ldots\ldots\; \frac{PALi-1}{2}^{\,ni} + PAP_{i-1}$$

$$\longleftarrow \qquad \longrightarrow$$

Figure 25: *Built up of the i-Pal at level i, being the final level.*

Because the structure of the final level is deviant and doesn't fit in the general model, the i-PAL lacks the transparent built up through its layer, that characterizes the o-PAL.

positioning the pair of i-PAL's relative to the o-PAL

As already stipulated in the description to Table I, there are always two i-PAL's within the sequentiality of the final level of a Christoffel sequence. Their rpp's lie one $\frac{PAL}{2}^{i-1}$ away (the one clockwise and the other counterclockwise) from the mirror axis. Figure 26a shows the position of the mirror axes within the model of the o-PAL.

The two PAP's at the end of each of the two i-PAL's can be understood in terms of a layered removal process, carried out on the string that is composed of the left or right inter-PAL_{i-2} (dependent on which of the two i-PAL we chose) and the direct adjacent lying inter-PAL $_{i-1}$. As example we take the i-PAL of which the mirror axis goes from top left to bottom right. The respective string on which we carry out a process of layered removal is indicated by a dotted frame in Figure 26a. Figure 26b presents the process of layered removal. By systematically decomposing a higher level palindrome in a number of lower level ones and at the end of each decomposing step stripping away the two palindromes of equal level at the two ends of the (remaining) string, ultimately only the palindromes P_1 and P_0 remain. These two are the ultimate 'breech' block (Figure 26c) within the circular presentation of the i- PAL_i , that by definition permanently must be deleted, except the $\frac{L}{2}$ elements at the left and right.

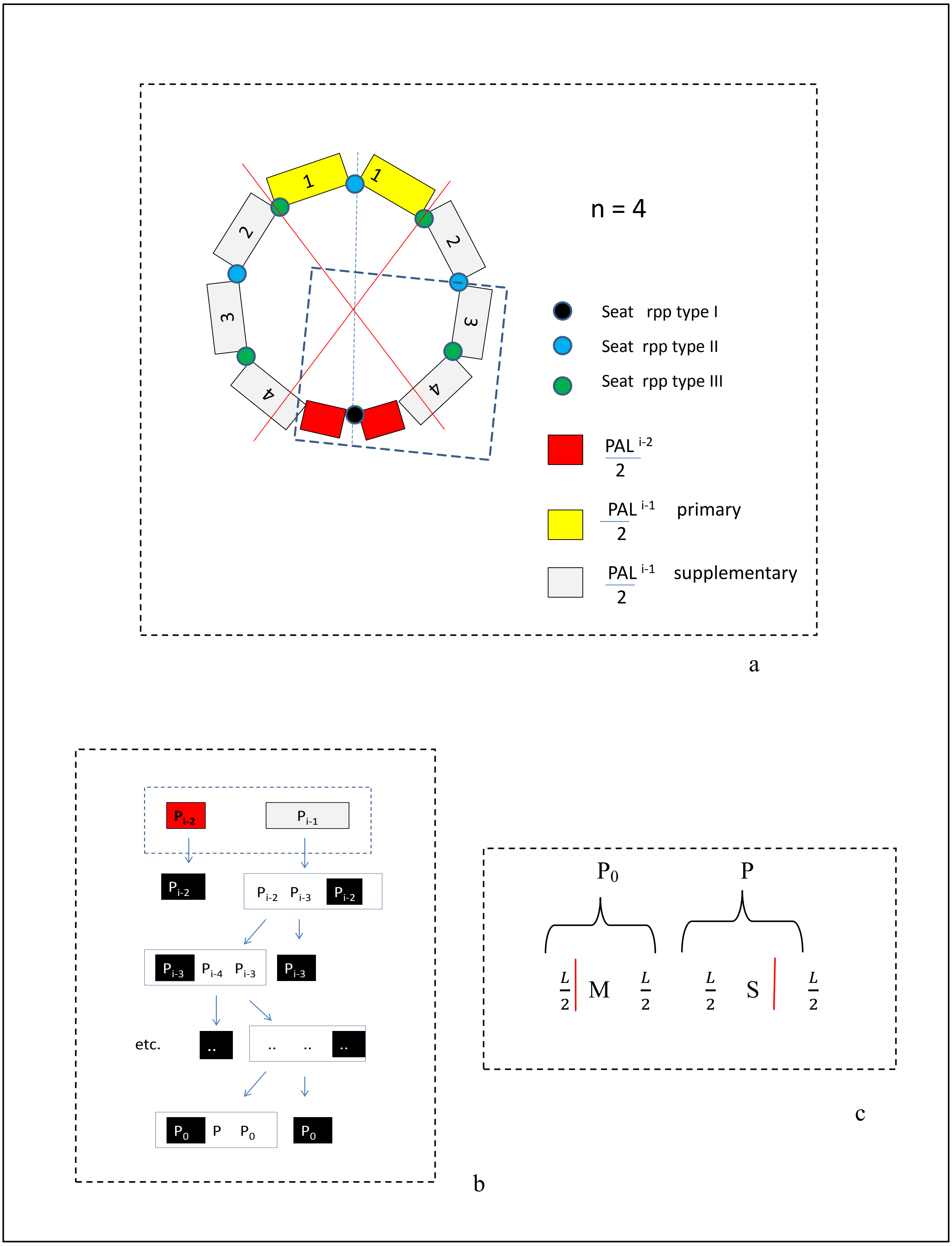


Figure 26*: Elucidation of the position of intra-PAL relative to the inter-PAL*

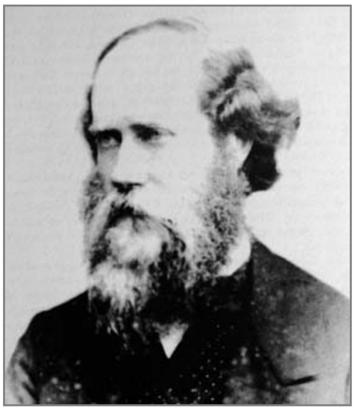
Janus head

In section 2.3.1 and 2.3.2 we discussed two quite different views on the structure of Christoffel sequences. From the one side you can conceive that structure in terms of a stratified alternation n and 1, resulting in an as evenly as possible spreading of L-events over S-events. This approach was discussed in section 2.3.1 The sequence structure is conceived here as being a-symmetric, but occurring in two different versions of that a-symmetry (Figure 17). Another way to look at Christoffel sequences is to see them as a layered whole of o-PAL's, as discussed in section 2.3.2. Christoffel sequences are like a Janus head[18] . They have two different faces. Each looks at us in its own enigmatic way.

2.4 Roots in mathematics.

number theory

It was only last year that I became acquainted with the ideas of mathematicians about Christoffel sequences. One can imagine how excited I felt when for first I became confronted with their ideas. In my fantasy explorers of new worlds must have felt in that way, when they put foot on shore there. My entrance was a paper of Caroline Series in this journal[19] The first thing that struck me was that she , though formulated in different words, considered the layered alternation of n and '1' as the main characteristic for this type of sequences. It was just what I had figured out on that dingy little road along the river Donau.
Caroline Series used a procedure of stepwise aggregation of the original sequence. In fact the procedure is the same as followed in the diagram in Figure 18.



In her paper she made reference to heritage of other mathematicians on this subject. So I learned that Elwin Christoffel was the first who had published about this type of sequences. His ideas were already presented in section 2.1. But just a year after this publication, another great mathematician of the 19th century, Henry John Stephen Smith, published about this type of sequences {3}. In his paper he expressed his pity that he wasn't acquainted with Christoffel's observations when doing his own research. And we may assume that this man really felt that sorrow. In comparison with Elwin Christoffel he was a very amiable man. When he died on February 9 of the year 1883, "his funeral procession in Oxford was a quarter of a mile long. This display of final respect was due mostly to Smith's immense personal charm and popularity with his contemporaries"[20].
Anyway, he was the first who noticed the wonderful layered structure of this type of sequences. And offered sound mathematical prove for the correspondence between the values n at the successive layers in the sequence and the partial quotients in the continued fraction. My knowledge of mathematics is too poor to understand it in all its details , but it was nice to see proved with so much sophistication what I always had felt to be true based on global intuition.

[18] Source picture: http://www.cybercauldron.co.uk/janus-and-january
[19] I hereby thank Dr C. Kraaikamp of the Technical University Delft (NL) who made me aware off it
[20] Even Smith's closest friends did not know of his mathematical achievements at the time of his death. Indeed, it now seems that a combination of his reluctance to self promote together with a serious miscalculation by mathematicians of his era led to his anonymity. Fortunately, mathematicians today are more aware of his manifold contributions to mathematics! *Mathematicians in History: Who was Henry John Stephen Smith? Baylor University, Department of Mathematics, Feb. 11, 2011*

In his treatise he uses an appearance of Christoffel sequences that we know from the movement of heavenly bodies as presented in Figure 6. He used the example *p/q*= 39/17. Writing for convenience P = 1/*p* and Q = 1/*q*, a line of unit length is divided in 39 pieces at the points 1P, 2P ,3P....and in 17 pieces at the points 1Q, 2G, 3Q... .The alternation of P and Q points in the final sequence has a layered structure. It is built up of a number of smaller sequences which themselves are built up of still smaller sequences. Higher and lower level sequences in this building up are represented by traverse lines:

P 2P Q | 3P 4P 2Q | 5P 6P 3Q | 7P || 8P 9P 4Q |
10P 11P 5Q | 12P 13P 6Q | 14P || 15P 16P 7Q |||
17P 18P 8Q | 19P 20P 9Q | 21P 22P 10Q | 23P ||
24P 25P 11Q | 26P 27P 12Q | 28P 29P 13Q | 30P |||
31P 32P 14Q ||| 33P 34P 15Q | 35P 36P 16Q|
37P 38P 17Q | 39P ||||

In essences his decomposition fully matches with the decomposion diagram in Figure 18. He too considered the L-events at the most basic level (in his notation the elements 7P ,14P, 23P , 30 P and 39P) as sequences of level 0.
Remarkably to note that neither Henry Smith nor Caroline Series Henry Smith appointed to the deviant pattern in the decomposition diagram when values 1 occur in the series of partial quotients. Henry Smith used [2, 3, 2, 2] as example and Caroline Series used [2, 4, 3, 2] .

combinatorics on words

Besides Henry Smith and Caroline Series more mathematicians have shown interest in Christoffel sequences. Especially in the last decades of the past century the field of 'Combinatorics onWords' laid hold on the subject. A disciplinewhich is strongly related to computer science. In their tradition of thinking they don't talk about 'Christoffel sequences' but 'Christoffel words'. A real explosion of publications came from this field. And dozens of these papers for more or less of their content were devoted to the subject of Christoffel sequences. In particular the following individuals have played an important role in raising the Christoffel sequences on the shield within this discipline: Jean Berstel, Valérie Berthé, Dominique Perrin, Aldo de Luca, M Lothaire (which is a pseudonym for a collective), Sebastion Aubbé, Francois Laubie, Chistophe Reutenauer, Jean-Pierre Borel, Christian Kassel, Eric Laurier. When I poked around in this field my initial reaction was: "Where the hell I am now ended up?" Their mathematical approach is very universal and for that reason has a dizzying high degree of abstraction. At the same time, being contradictory to that, they sprinkle abundantly with technical jargon to be able to describe and analyze all sorts of phenomenon in strings of letters[21]. But, after some habituation I could make some sense out of their way of reasoning.

[21] Prefix and suffix, conjugates, lexicographic order, standard factorization, palindromic closure, etc..

Those guys (indeed, all are men) are not primarily interested in the layered structure of Christoffel sequences. A layeredness that quite transparent can be traced back to the stages in the development of the continued fraction of the respective *s/l*-ratio. Central in their reasoning is the idea that the collection of all Christoffel sequences is a so called 'free homoid'. That means that each of these sequences can be composed out of a restricted set of basic elements by means of a binary operation. In the case of Christoffel sequences the basic set of elements are the two letters and the binary operation is the 'concatenation'. Smaller sequences coming forth of those binary operations, can themselves also be directly concatenated tot larger sequences having the character of Christoffel sequences. The tree-structure in Figure 27 gives an expression of this evolution-like process. From top to bottom, at each vertex there are two possibilities: left and right concatenation.

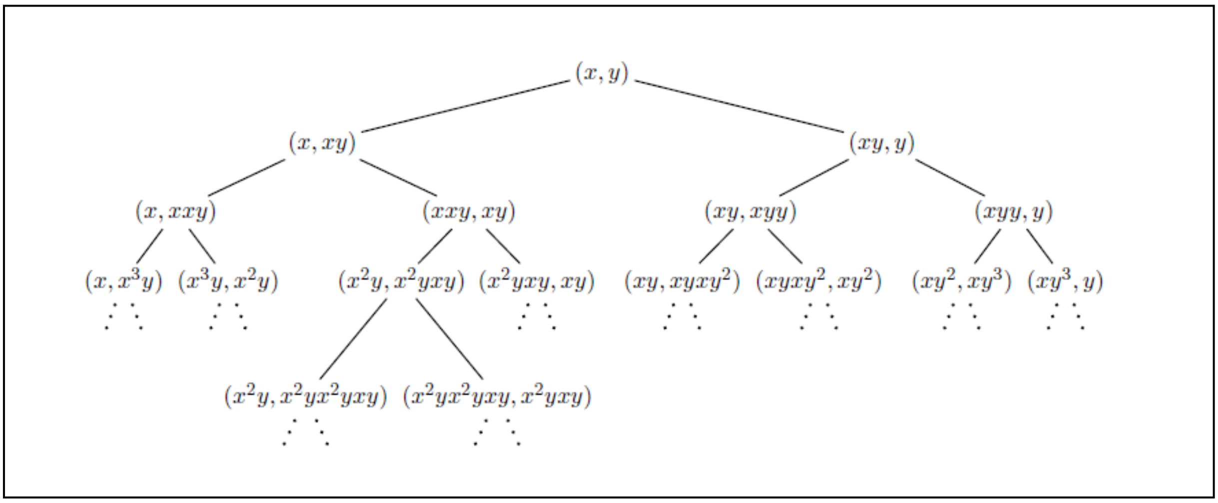


Figure 27 ***:*** *Tree structure showing the evolution in Christoffel words*

better to put the new wine in old wine skins*!*
It seems to me that the strength of this new and explosive growing discipline is that they are able to explain the phenomenon of Christoffel sequences in quite universal terms. But did their concepts add to the conceptual heritage of Henry Smith, who made explicit the beauty and elegance of Christoffel sequences. As far as I can see the answer is "no". In the hands of those 'Combinatorics', Christoffel sequences have loosed their luster and magic, being placed in a tradition of thinking about computer storage and computer calculation. So while the Bible warns us not to store new wine in old wine skins, in this case my feeling is: Let the new wine for Heaven's sake ripen in old wine skins.

Source : https://en.wikipedia.org/wiki/New_Wine_into_Old_Wineskins

3 Christoffel sequences in Escher'garden

Now the reader may have some idea of the wonderful characteristics of Christoffel sequences Next thing to do is answering the question why these sequences are so tightly interwoven with plane symmetries? The answer to this question begins with some notions about the essence of plane symmetries.

grids of parallelograms

Plane symmetry is about the spreading of points that are equal[22], in an infinite plane. Every point in a plane symmetry has infinitely many 'equals'. When we choose an arbitrary group of such 'equals' and mutually connect them by lines that run in the two basic shift directions, we get a lattice of parallelograms (Figure 28). The connected points are called the lattice points and the structure demarcated by one parallelogram is called a repeat unit[23]. Within a repeat unit every point is unique[24]. The 'equals' of these points are located in the other repeat units. This principle is the same in all 17 plane symmetries[25].

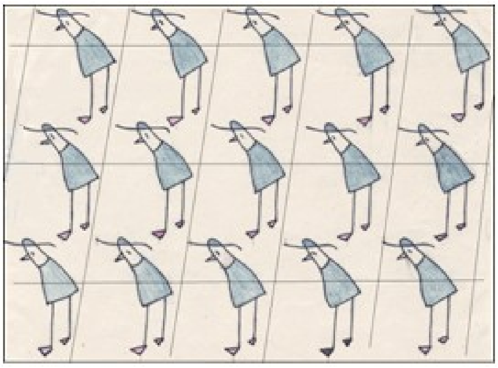


Figure 28 : *Lattice of parallelograms.*

direction

By drawing a straight line between two equal points that coincide with vertices of the grid, we introduce a direction within it. Because of the symmetry constraints, that line segment must repeated. Firstly it becomes a repeating building block within a line that runs into infinity (Figure 29a). And this infinite line itself must be repeated ad infinitum. That is: it must be drawn through all vertices of the parallelogram grid (Figure 29b). The resulting pattern is an infinite number of stripe paths, adjacent to each other, running into infinity while traversing a grid of parallelograms.
The direction of the line segment of which this structure is built up by repetition ad infinitum, is specified by the relative position of the points that form its beginning and ending[26]. That relative position can be quantified in terms of a Cartesian coordinate system. One point is taken as origin and the two coordinates coincide with the two translation directions of the lattice, whereby the sides of the parallelogram deliver the units of distance. We denote the resulting direction as s/m . Figure 30 illustrates the example 2/5. The stripe paths divides the two sides of the parallelogram in a different number of pieces. In the example, the side that coincides with translation direction I is divided in m pieces (let's call them 'm-pieces') and that in translation direction II in s pieces (' s-pieces). The numbers m and s are called Miller indices. In the tradition of crystallography the direction of crystal faces is defined in terms of these indices.

[22] They have an equal environment and an equal orientation of that environment.
[23] As far as that structure coincides with the lattice or with the lattice points, it is shared with other repeat units.
[24]On the border each point occurs two times but is shared therewith another repeat unit.
[25] In higher order plane symmetries that shape becomes upgraded to the shape of a diamond, rectangle or square in plane symmetries with higher symmetry levels (dependent on the number of symmetry operators; foldness of rotation axes).
[26] We reserve the number l for the diagonal line that split the OTRU in 2 OBRU's

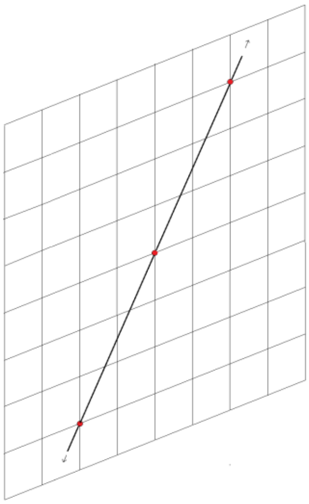


Figure 29: A line running in a certain direction through a plane symmetry can be seen as built up of line segments that have their beginning and ending in a pair of equal points.

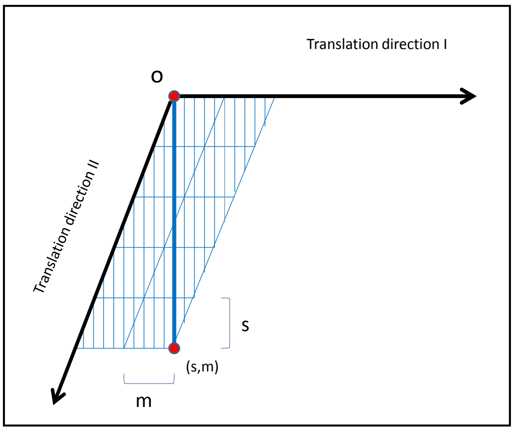


Figure 30: Direction specified in terms of Miller indices.

Christoffel sequences

Imagine someone walks on such stripe path, traversing one parallelogram after the other, on the way to infinity (Figure 31). Some of these parallelograms she enters via the side that is divided in m pieces(we call them 'm-pieces'), and some via the side that is divided in 's-pieces' (we call them 's-pieces'). Let's call the passing of an m-piece an 'M-event' and the passing an s-piece an 'S-event'. Walking over the stripe path into infinity the person is generating a Christoffel sequence built up of m M-events and s S-events. In the example the period of that sequence is MMMSMMS. Because we are dealing with plane symmetries, the S- and M-events can also be conceived in terms of the spatial transformations involved (Figure 32). Every time an M-event takes place, the parallelogram entered can be conceived as arising from the parallelogram left by a translation in direction I. And every time an S-event takes place, parallelogram entered can be conceived as arising from the parallelogram left by a translation in direction II.
The preceding was very basic: a stripe path in a certain direction, traversing the repeat units of a plane symmetry, evokes a Christoffel sequence. Stripe paths can also evoke complemented Christoffel sequences. Therefore we need to look with more nuance at the 17 plane symmetries.

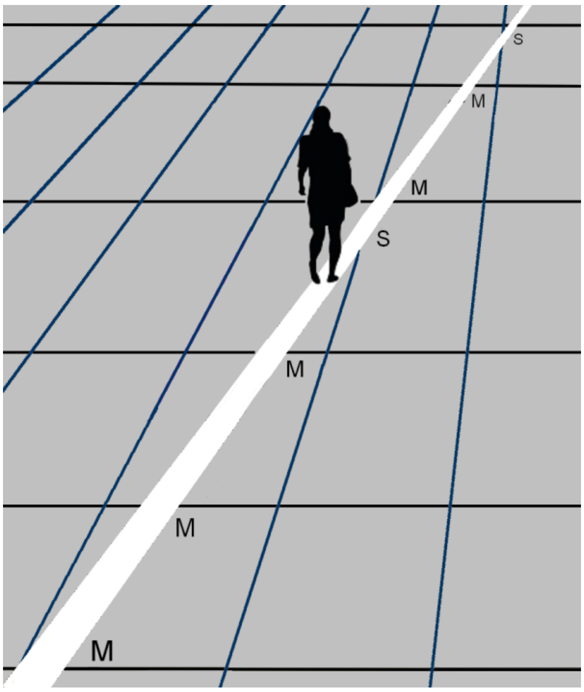


Figure 31: Woman, generating a Christoffel sequence by alternately traversing M-pieces and S-pieces of the outline of the repeatunit.

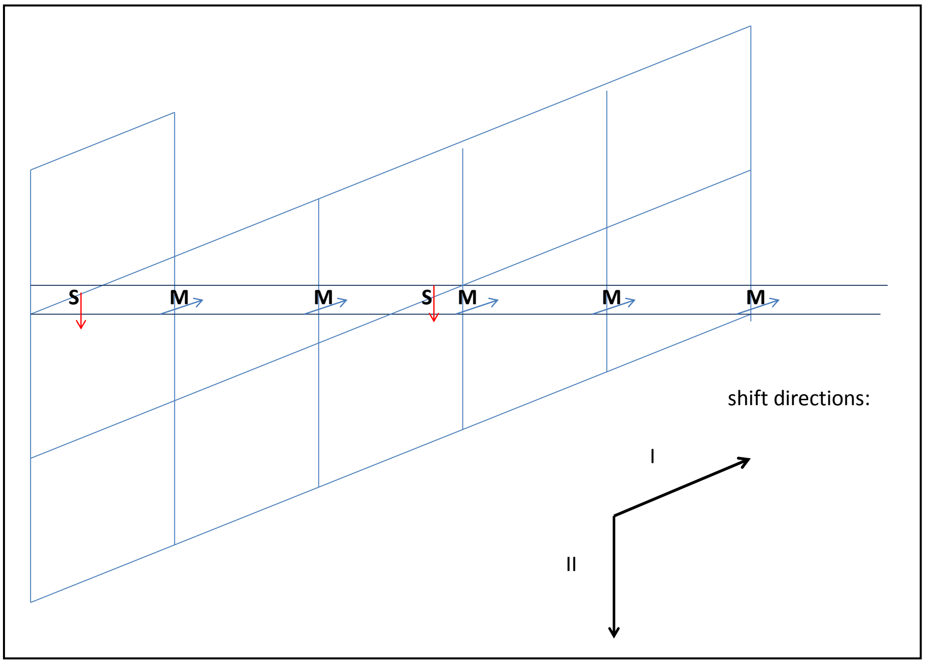


Figure 32: *The S- and M-events can be conceived in terms of the spatial transformations involved.*

differences between the seventeen

Given the main structure of repetition in every plane symmetry, there is substantial difference in how symmetrical operators are active in each of them. We discriminate 4 possible symmetry operators: translation, reflection, rotations and slide reflection. Translation in two directions is standard present and determines the form of the lattice. The other three are optional. Doris Schattschneider [4] once gave an imposing overview of how the 17 plane symmetries differ from each other in terms of these operators (Figure 33).

In this paper not all 17 plane symmetries are relevant. We consider only those in which rotations occur beside translations. There are 4 of them. In the overview in Figure 1 they are enlisted red or yellow. They are called p2, p3, p4 and p6, the numbers referring to the highest 'foldness' of the rotation axes showing up each of them. sequences.

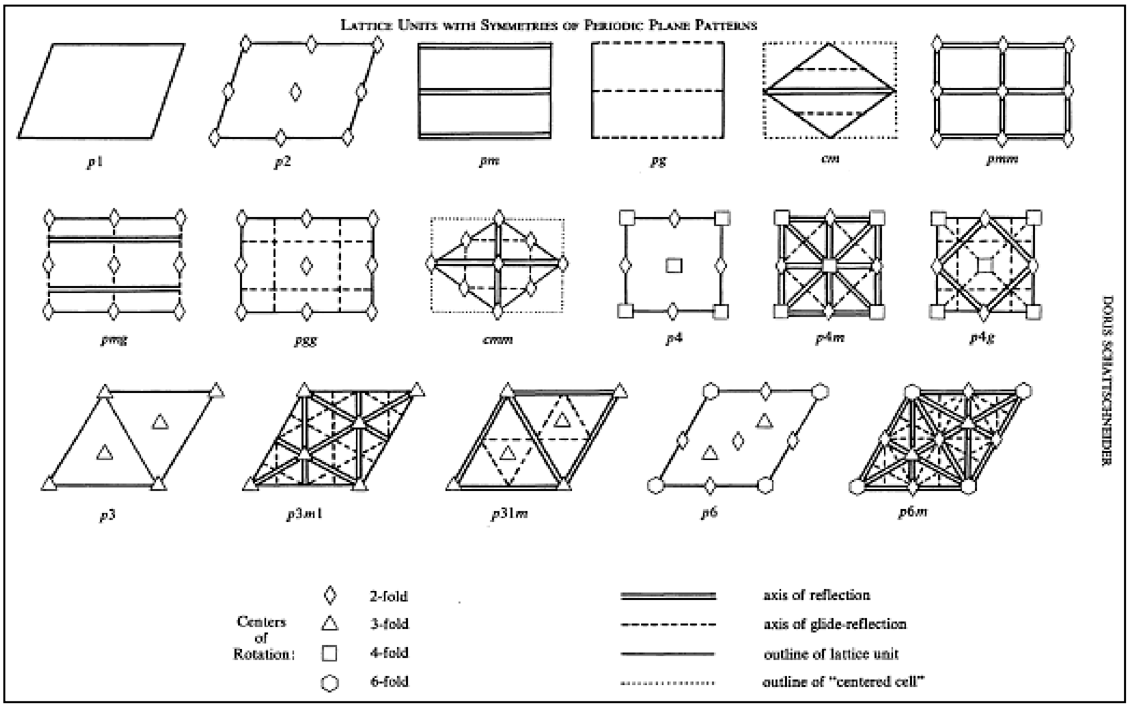


Figure 33 ***:*** *Overview of the presence of symmetrical operators in each of the 17 plane symmetries.*

OTRU's and OBRU's

When other symmetrical operators beside translation are active in a plane symmetry, more 'motifs' show up within the outline of one repeat unit. Figure 34 shows an example in p2. The repeat unit can be divided in two sub units (gray and yellow), each containing the motif, but in a different orientation in space. The example shows that there are two levels in the repetition-structure of a plane symmetry that contains other symmetrical operators beside translation: the level of the OTRU's (Orientation Transcending Repeat Unit) and the level of the OBRU's (the Orientation Bound Repeat Unit). The number of orientations in which OBRU's show up, depends on the number and type of optional symmetrical operators. In the plane symmetries we focus on in this paper, p2 , p3, p4 and p6, the number of orientations equals the highest 'foldness' among the different types of rotation axes.

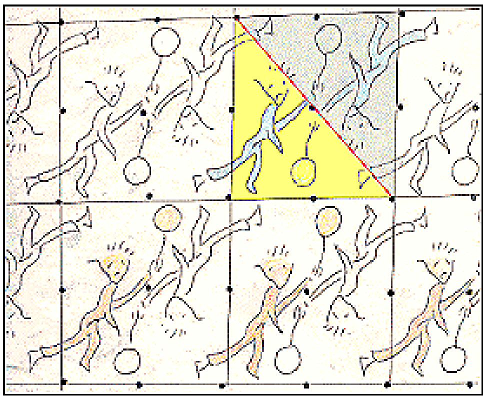


Figure 34: *Difference between repeat unit and primitive repeat unit.*

In the example the highest foldness is 2, so the OBRU's show up in two orientations. You can choose the shape of the OBRU's rather free[27]. In the case of p2 there's no need to restrict it to the triangle. The only requirement is that the symmetry operators that are active beside the translation in two directions, which in our case are the different types of rotation axes, lie on the

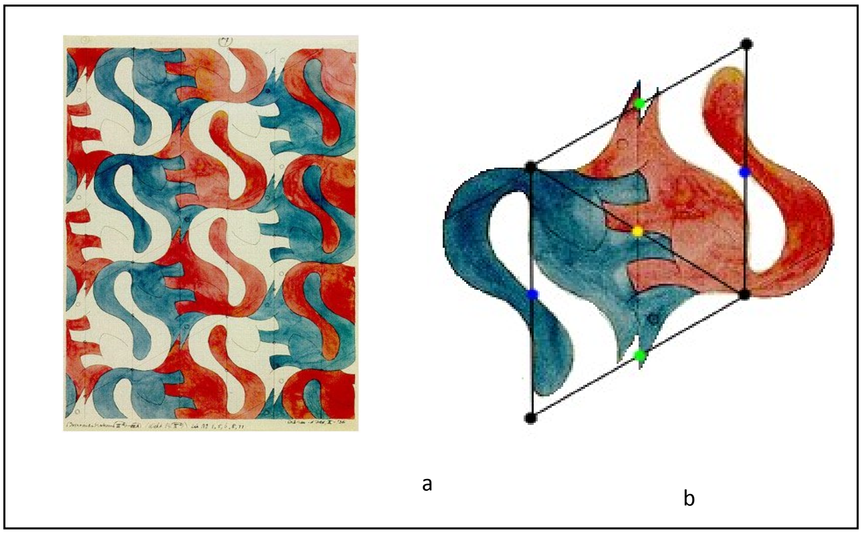


Figure 35*: Esscher's picture of squirrels is a plane symmetry of type p2*

outline of the OBRU. An appealing example of such a free chosen shape in p2 we find in Escher's picture of squirrels (Figure 35a). Each of the animals represents an OBRU[28]. As may be clear , all 4 types of rotation axes are represented on the contour of the squirrel. We can convert the shape of the squirrel into that of a triangle, leaving the condition in tact that all 4 types of rotation axes are presented on the outline of the area [29] (Figure 35b).

[27] Also the shape of OTRU's can be chosen rather free. And the outlines of the different oriented OBRU's needn't to fit within the outline of an OTRU.

[28] If we neglect the different colors of the animals.

[29] In the trangle as OBRU, the orange rotation axis reoccurs three times on the outline. But this axis is shared with more OBRU's than the blue, yellow and red one, that are shared with only one other OBRU's

complemented Christoffel sequences

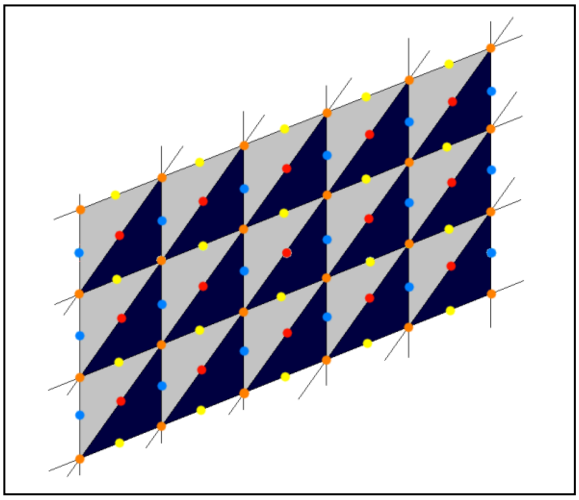


Figure 36: *Tessellation of OBRU's.*

With the aid of OBRU's we can quite simple generate complemented Christoffel sequences, especially in p2. Within p2 each OTRU can be divided in two OBRU's by drawing a diagonal in it, the OBRU's having the shape of a isosceles triangle[30]. Figure 36 shows the resulting tessellation, including the dispersion pattern of the rotation axes. There are 4 different types of 2-fold rotation axes. By definition an exemplar of all 4 types is present on the outline of one OBRU's.
A stripe path running through such a tessellation of OBRU's generates a complemented Christoffel sequence. Each of the three sides of each OBRU is intersected . The ℓ-side is intersected ℓ imes,, the m-side m times and the s-side s times(Figure 37b). So three different types of events takes place: S- events, M -events and L-events. Figure 37a shows one period of this complemented Christoffel sequence.
Here too, the three events can be described in terms of spatial transformation. Every time an S-event takes place, the OBRU entered can be conceived as arising from the OBRU left by a 2-fold rotation around the yellow axis that lies on the s-side that is shared by both OBRU's. In the same way an M-event is associated with a 2-fold rotation around a blue axis and an L-event with a 2-fold rotation around a red axis.
The starting and ending point of the period in Figure 37a are chosen in such way that it manifests itself as an o-PAL. Of the two rp's of this o-PAL lies in the mid . The other lies at the beginning or ending of the period. This one best can be presented as lying half at the beginning and half at the end . This rp-half then is the seat for half an M-event. It means that the respective M-event is shared by two adjacent periods. The rp in the mid divides the o-PAL in two symmetrical halves, of which one is colored green and the other blue. It is the seat of two halves of the letter L. Figure 37c explicates the elementary o-PAL halves of which these two o-PAL halves are composed. Let's have a closer look at one of the two (Figure 38a). It is built up from ℓ basic o-PAL halves, which are of two different types, each of these occurring in two mirror images : $\frac{L}{2}\frac{S}{2}$ with its mirror image $\frac{S}{2}\frac{L}{2}$ (both colored blue) and $\frac{L}{2}\frac{M}{2}$ with its mirror image $\frac{M}{2}\frac{L}{2}$ (both colored yellow). Pair wise they form o-PAL wholes[31]. All these ℓ halves are stapled in one OBRU (Figure 38b). Each of the equilateral stripe segments in it is an basic o-PAL halve. The three rotation axes in the mid of the three sides of the OBRU determine how they are connected together. So all in the information about how the o-PAL halve is composed of basic o-PAL halves is stored in one OBRU. In a couple of basic o-PAL halves being each other's mirror image, the palindrome structure in the ordering of letters halves matches with a geometrical rotation symmetry in the intersection angles (Figure 38c).

[30] The length of the sides of the OTRU are supposed to differ two by two, related to the difference in shift distance on the two shift directions.
[31] When both adnl are odd, one basic o-PAL has no mate in its own o-PAL halve, but is connected to one in the other o-PAL halve.

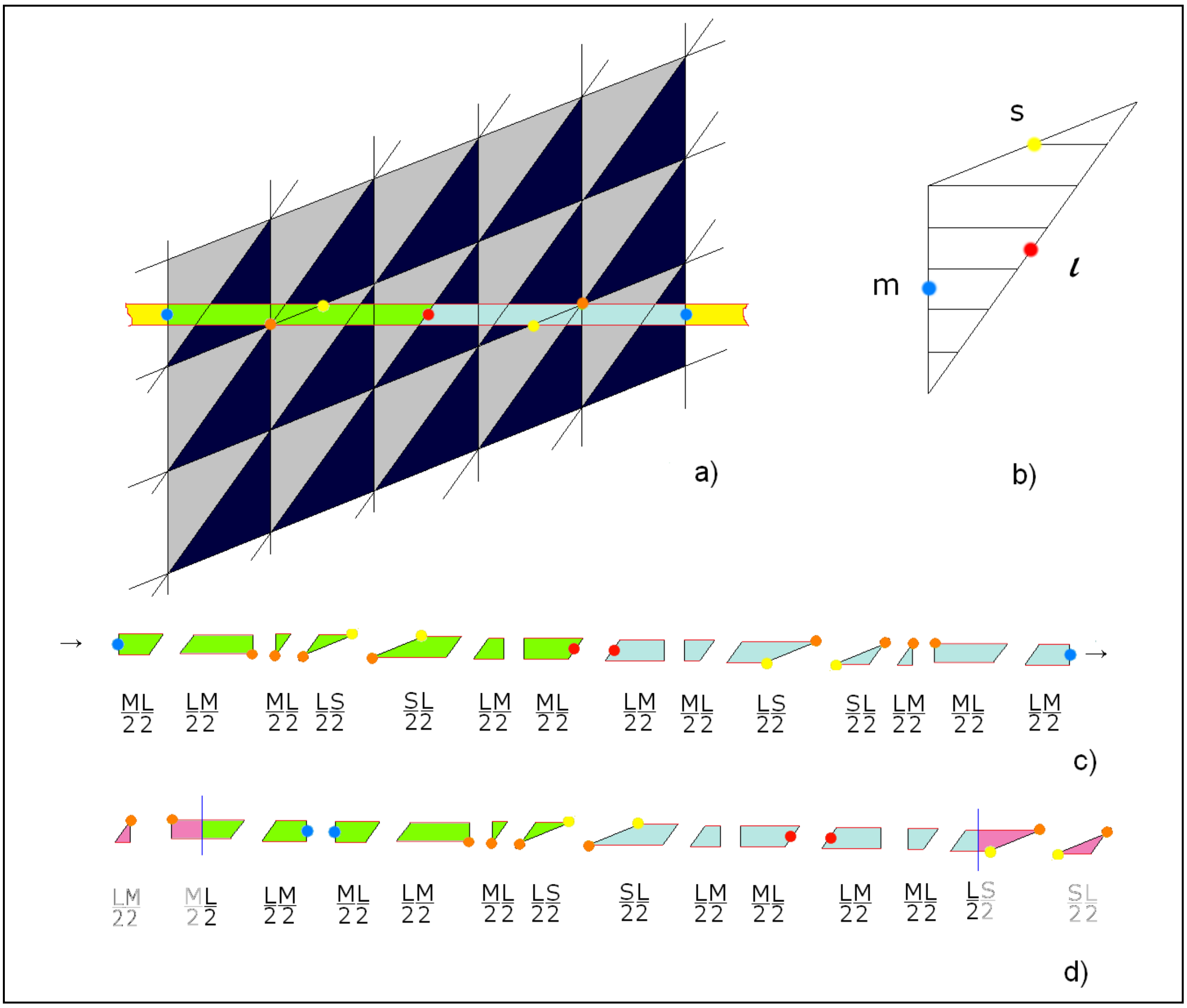


Figure37 ***:*** *On its way to infinity a stripe paths generates complemented Christoffel sequences, by alternately traversing the different sides in the outline of OBRU's.*

As stated before, the starting and ending point of the period in Figure 37a are chosen in such way that it manifests itself as an o-PAL. But you can as well choose them in such way that the period manifests itself as an i-PAL. Figure 37d shows that rearranged period. Three letters, S,L,M, marked in gray color, must be removed to get two symmetrical halves in the ordering of elements. The only rpp in the i-PAL is represented by an S-event and coincides with a yellow rotation axis. Picture 37 makes clear that especially the rotation axes that lie in the mid of the sides of the OBRU coincide with rpp's: one with the rpp in the i-PAL and the other two with the rpp's in the o-PAL.

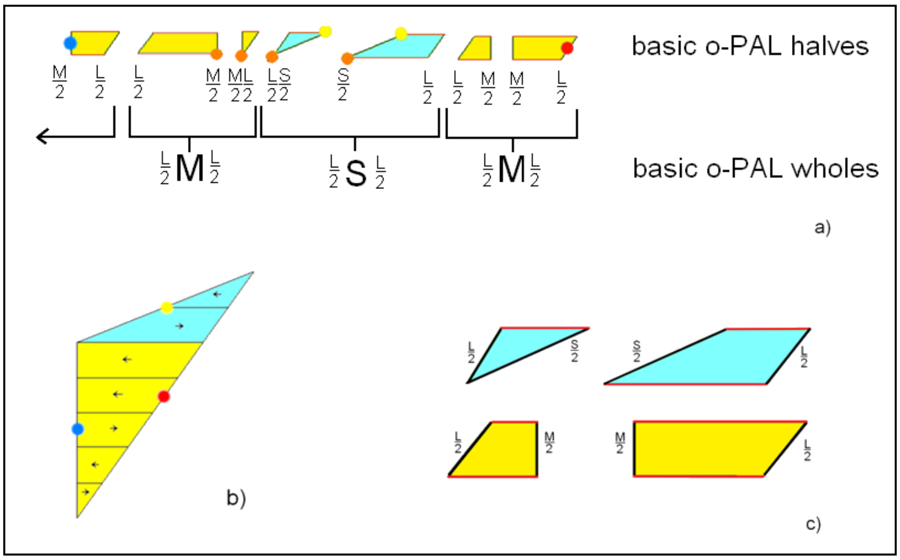


Figure 38: *Each is built up from l basic o-PAL halves, which are of two different types, each of these occurring in two mirror images :* $\frac{L}{2}\frac{S}{2}$ *with its mirror image* $\frac{S}{2}\frac{L}{2}$ *(both colored blue) and* $\frac{L}{2}\frac{M}{2}$ *with its mirror image* $\frac{M}{2}\frac{L}{2}$ *(both colored yellow). Pair wise they form o-PAL wholes[1]. All these l halves are stapled in one OBRU (Figure 22b). Each of the equilateral stripe segments in it is an basic o-PAL halve.*

Mathematical evidence for Christoffel's algorithm

Practice learns that a stripe path running through a tessellation of OTRU's always generates a Christoffel sequence. But can we make this evident on logical grounds.Yes we can! And when it is evident for a tessellation of OTRU's it is also evident for a tessellation of OBRU's , because in that last case an L-event is inserted after every M-event and after every S-event. To make it evident we need to focus only on one OTRU with two OBRU's falling within its outline. It contains all the stripe path segments of one period and contains all the necessarily information about the order in which they are connected. Figure 39 shows as an example the OTRU of which s/m = 3/7. The 'entrances' of the stripe path segments are indicated by blue letters; the 'exits' by pink letters. The letters indicates the order in which the segments connect to each other. The pink exit g for example connects to the blue entrance g in the direction of the arrow. On the diagonal we have numbered the segments. Starting at the exit of segment 0, we go to segment 3 , which implies that we have added 3 to 0. Then we go successively to segment 6 and 9, which also imply the addition of 3.

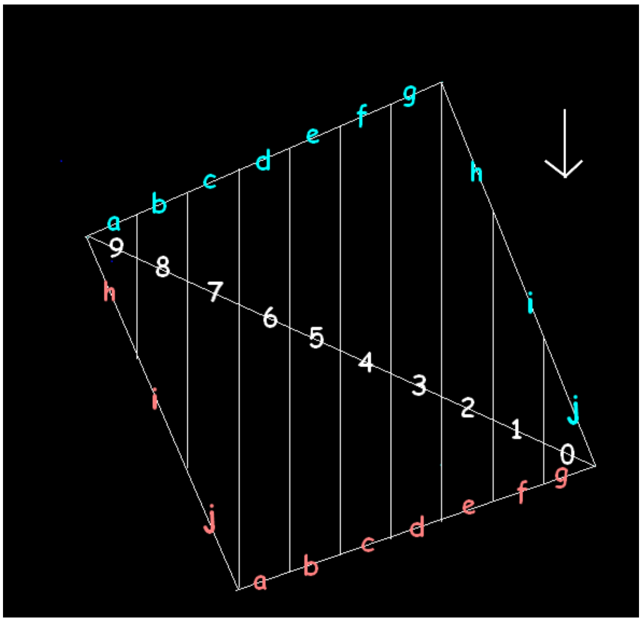


Figure 39 *: Christoffel's algorithm projected on a repeat unit.*

Then we go to segment 2, which implies an addition of 3 followed by a subtraction of 10, because the modulus was exceeded. Etc.. Each time the modulus is exceeded after an addition, the entrance side the segment traverses a s-side in the outline of the unit. In the other cases the entrance side traverses an m-side in the outline of the unit. So after one period we have got the sequence MMMSMMSMMS.

generating beauty
The preceding illustrates that Christoffel sequences quite naturally show up in plane symmetries of type p2. But the esthetic quality is not very high. Far more beauty can be realized when we use a more a more complex algorithm that's applicable p3, p4 or p6. It results in those wonderful stripe paths which I discovered in the summer of 1969.

4 Spirals

4.1 Generating stripe paths by regular grid division

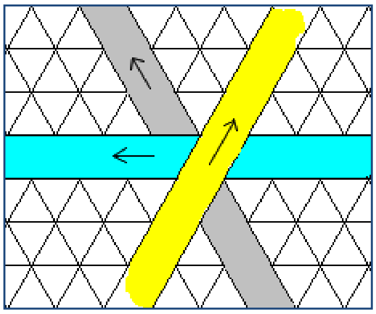


Figure 40: Grids of triangle can be conceived as built up of stripe paths that run into infinity in three different orientations.

No geometric appearance expresses the wonderful characteristics of Christoffel sequences in a more appealing way than the stripe paths that I by accident discovered in 1969. They can be generated in grids of squares or triangles. Such grids can be seen as built up of straight stripe paths that run into infinity in three (triangles) or two (squares)orientations (Figure 40). If you divide a grid in equal areas and select only one of the two or three orientations in which the pieces of stripe path run in each area**,** automatically stripe paths show up which have the character of closed circuits (Figure 47).

units of grid division

Let's call the areas in which a grid is divided 'units-of -grid-division'. I designed three different types: one for square grids and two for triangle grids. Figure 41 show them in the example *l*/m/s = 2/3/5. The line segments that constitute the outline of the areas coincide with lines of the grid or run diagonal through the squares or diamonds (in the case of a grid of triangles) of the grid structure. By realizing regular grid division by means of these units, the grids acquire a specific type symmetry. The symmetry of the square grid becomes p4 and the symmetry of the triangle grid becomes p6 or p3[32] respectively. In the three types of plane symmetry, three different types of rotation axes occur that lie on the outline of the units[33]. Each of the three axes is connected to an 'own' couple of line segments of equal length. The three length values, large (*l*), medium (*m*) and small (*s*) are partly mutually independent (*l* = *m* + *s*) and relative prime. Factually the couples of line segment connecting the rotation axes are *deflection lines*, brought into the grid (see description in the frame below).

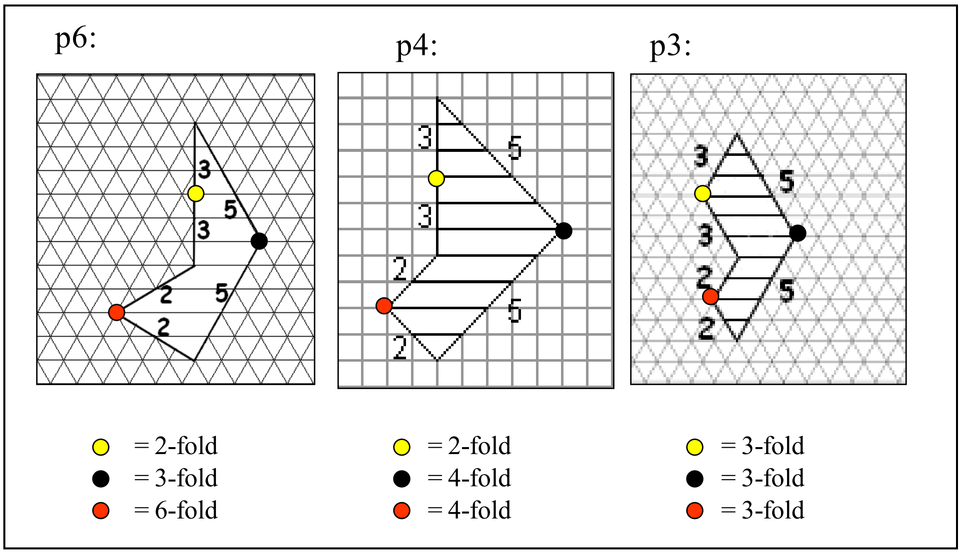


***Figure 41** : Units-of-grid-division in p6, p4 and p3*

[32] Originally the symmetry levels of the square grid and the triangle grid are respectively p4mm and p6mm. The mirror axes disappear and in the case of p3 also the highest foldness of rotation axes decreases from 6 to 3..
[33] The units-of-grid-division are OBRU's, so of each of the three types of rotation axes lies a representative on the outline of the unit.

A deflection lines ' causes' a stripe path to change its course (Figure 42). Hitting a deflection line, the entering and the exiting path have the same angle with it, so the path maintains its width after a change in its course. The part of a deflection line which is traversed by the path is considered as *the unit of length.* So the length of deflection lines always can be expressed in integers.

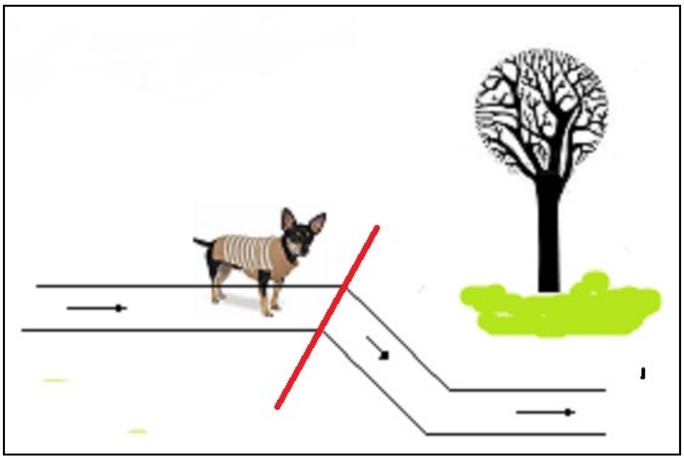


Figure 42 *: Deflection lines*

Already my whole life I work with these three variants. The story how I came to them is quickly told. The unit in the middle was derived from a pattern of Greek crosses, that I found on the robes of the Cappadocian Church fathers (Figure 43) on a Byzantine icon. It was this pattern which inspired me to make that wonderful journey through Alice's wonderland. The unit at the left is a direct conversion of the unit in the middle to symmetry type p6. The stripe paths that emerge from specific this unit-of-grid-division excel in elegance and beauty, as already was shown in Figure 4. Still I am most proud of the unit of grid division at the right, because of the pureness and mystery of the resulting patterns and the cultural background of the stripe path of which it was derived. I derived it from one of the most elegant ornamental art expressions in Chinese culture: the three legged cross (Figure 44). Owen Jones already presented it in his Grammar of Ornaments. The s/l/m fraction is 6/6/12. The respective number must be divided all by 6 to let them be relative prime. Beautiful variations in terms of color mixture can be realized with the aid of this unit of grid-division (Figure 45).

Figure 43*: Cappadocian Churchfathers.*

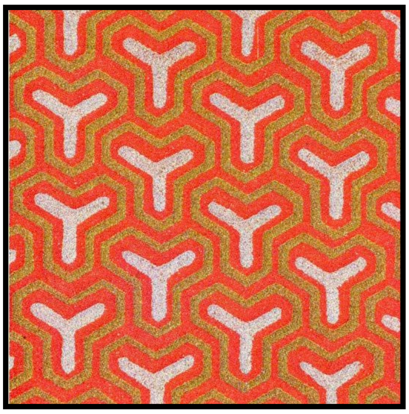


Figure 44*: Chinese decoration with three-legged-crosses*

Figure 45: *Beautiful color mixtures can be realized with the unit-of-grid-division in p3.*

I have always considered these three units-of-grid-division as the primeval ones. There is some ground for that. They can be derived from the three most basic tessellations in which regular polygons show up of which the size can be varied mutually independent (Figure 46). For p6 these polygons are a hexagon and a triangle, for p4 they are two squares and for p3 they are three triangles[34]. In these tessellations we can select OBRU's that are built up of sections of the respective polygons. These OBRU's are the units-of-grid-division we just described.

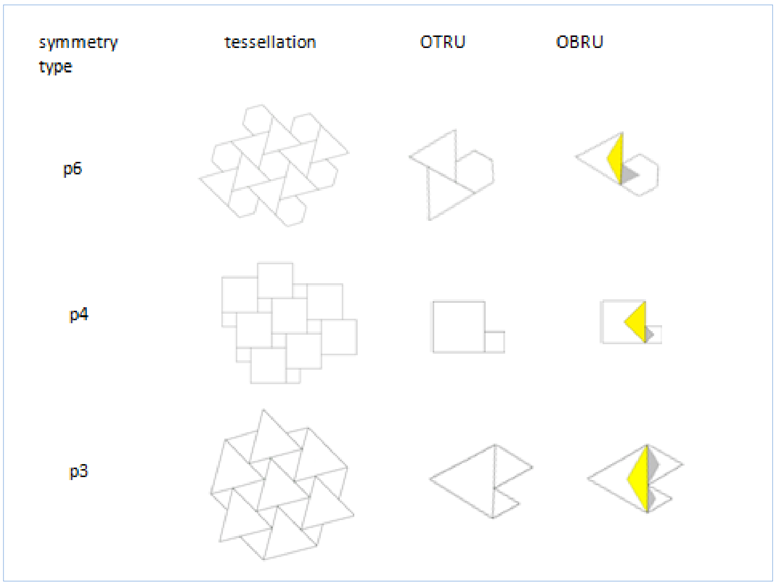


Figure 46: *tessellations built up of polygons of which the size can be varied mutually independent (in p3 two of the three)*

stripe paths generated by regular-grid-division

As noted before, when you select but one orientation per unit of grid division [35], automatically stripe paths arise which are closed circuits. To these stripe paths complemented Christoffel sequences are inherent. For every ratio *s/m/l* an unique stripe path emerges[36]. Like the structure of complemented Christoffel sequences in general (see section 2), the structure of all these unique stripe paths can be grasped in terms of the partial quotients of the ratio s/m. Figure 47 illustrates the example s/m/l = 2/3/5 in p6[37]. All stripe paths which can be generated in this way have the character of a closed circuit that consists of an number of protrusions which are symmetrical arranged around a rotation axis in the center. In each of the protrusions the stripe paths runs in two directions with at the end a reversal in the direction (at the point of the white arrow). So every stripe path has an inner and outer border. The outer border can be conceived as a concave polygon. Of the three types of rotation axes always two lie at the outer border of the stripe path and one at the center of its inner border. Meandering through

[34] In this last tessellation only two of the three triangles can be varied mutually independent as to their size.
[35] The one in which of none of the stripes a border line coincides with a border line of the unit of grid division.
[36] The ratios 0/1/1 generates the hexagon.
[37] Examples of stripe paths generated in p4 and p3 are presented in Figure 5 and in Figure 43 respectively.

the plane the stripe path alternately crosses deflection lines of length *s, m* or *l*, generating (in the direction of the arrow) a sequence with period LMLMLSLMLS . The letters S, L and M represent angles of 60º, 120º and 180º respectively. The period repeats itself *f* times, where *f* stands for the 'foldness' of the rotation axis in the center of the stripe path.

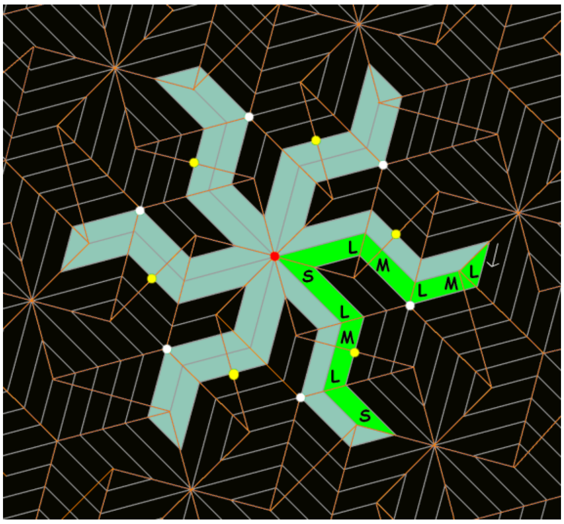


Figure 47: *When one orientation is selected per unit of grid division, automatically a stripe path arises.*

Stripe paths are Christoffel sequences

That the generated sequence is always a Christoffel sequence not only appears from practice but can also be deduced in a direct way from the 'working' of the unit of grid division. Not easy however. It requires a feat in number symmetry .The reasoning is as follows. A generated stripe path traverses always more than one unit of grid division. We can represent all these traverses within one unit of grid division, as shown in Figure 48 for the example 3/7 in p6. The unit of grid division (gray surface) is built up of 2 equilateral triangles of different size, lying opposite to each other. An 'compensatory' line segment is drawn where the bases of the two triangles do not overlap. In the point between the two equilateral sides of the two triangles a rotation axis is situated. The two axes of rotation are of different type. In the mid of the compensatory line lies the third rotation axis , dividing it in two equal parts. The length of the two pairs of equilateral sides are l and s respectively; the lengths of the two parts of the compensatory line segment each are m. In the example ι = 7, m = 4 an s = 3. Within the unit of grid division pieces of stripe path are running. They are connected by arc wise running pieces outside the unit. The result is a closed circuit. The connection pattern is such that in the resulting circuit the pieces of stripe path are connected to each other in the same consecution as in the real stripe path. Through the base of the largest triangle we draw an imaginary line that partly coincides with the compensatory line that is already present there. That imaginary line is crossed by 7 upward running pieces of the stripe path and 7 downward running pieces. Now we are ready to implement Christoffel' s algorithm. We number the set of downward running strokes from 0 to ι - 1 and the set of downward running stokes from s to s + ι - 1 in opposite direction. In the example the numbers are going from 0 to 6 and from 3 to 9 respectively. With the application of this numbering, Christoffel's algorithm for the generation of two-letter sequences (S and L) is implemented . Starting at 0:

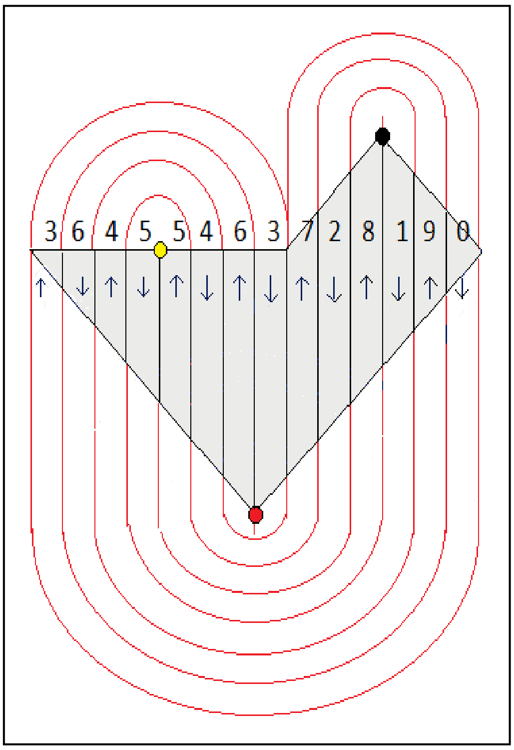


Figure 48*: The course of stripe path s/l= 3/7 depicted in 1 unit-of-grid-division.*

- Every time you change symmetrical from downward to upward in the large triangle, you add 3 to the preceding number, which is an L-event
- Every time you change symmetrical from upward to downward in the small triangle, you subtract 7 from the preceding number, which is an S-event.

two types of palindromes

There are two types of palindromes present in every Christoffel sequence: two intra-PAL's and one inter-PAL's. So they can be find also in our stripe paths. Primary we find them the course of the stripe path in its meandering through the pane, intersecting there alternately L(ong) M(edium) and S(hort) deflection lines . But we can see them, in a more transparent and explicit way, in the inner and outer border of the stripe paths. Figure 49 illustrates the example *s/l/m* = 2/3/5. The white letters represent the i-PAL that lies on one of the six possible pairs of dendrites within the inner border. It repeats itself f times, whereby at each time one of the two $\frac{i-PAL}{2}$'s overlaps with a $\frac{i-PAL}{2}$ of the preceding pair. So the inner border is quite naturally divided in f $\frac{i-PAL}{2}$'s. All these f $\frac{i-PAL}{2}$'s deal the internal axis of rotation as seat for their rpp halve, that is represented by the letter halve $\frac{S}{2}$.
The blue letters represent the o-PAL. It lies on the outer border and repeats itself *f* times. In the mid of it lies a rotation axis that is the seat for one of the rpp's. The axis quit naturally divides the inter-PAL in two $\frac{o-PAL}{2}$'s .

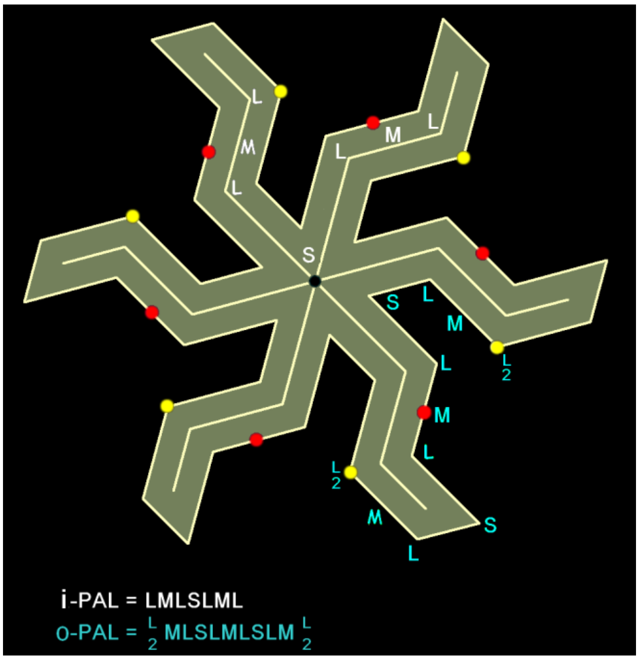
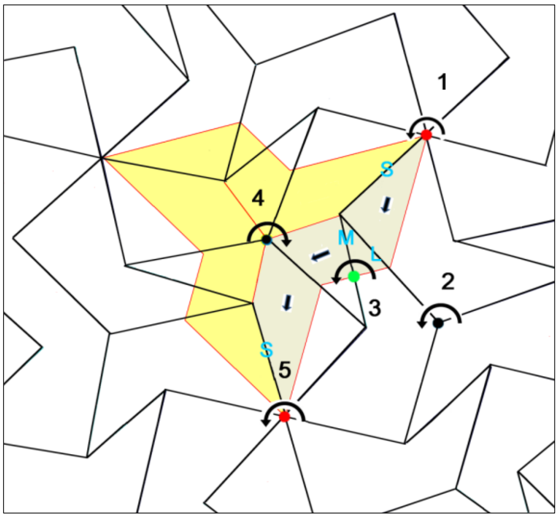
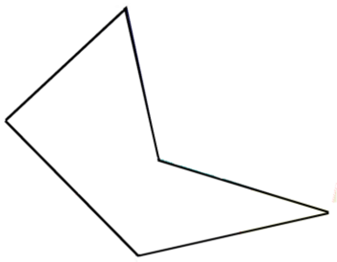


Figure 49: *intra-PAL and inter-PAL in respectively the inner and outer border of stripe path s/l/m = 2/3/5*

Christoffel sequence as a sequence of rotations of an OBRU around its rotation axes

Like the Christoffel sequences generated by the stripe paths that run straight into infinity (section 3) , the Christoffel sequences we discuss here can be seen sequence of rotations of an OBRU around the rotation axis that lie on its outline. Let's have a closer look at the example is *s/m/l* = 1/1/2 in p6





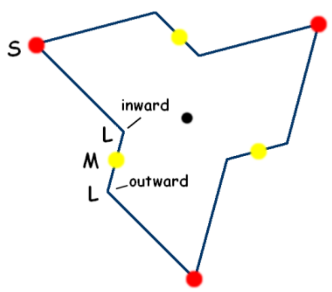
OBRU

Figure 50 : *Christoffel sequences can be seen as a sequence of rotations of an OBRU around the rotation axis that lie on its outline.*

(Figure 50). The gray area represents one period in the respective Christoffel sequence : $|\frac{S}{2}\frac{S}{2}$ L $|\frac{M}{2}\frac{M}{2}|$ L $|\frac{S}{2}\frac{S}{2}|$. The respective S-, M-, and L-events are generated by the stripe path in successively traversing deflection lines of length *s, m, and l.* At these events of traversing the entered OBRU can be seen as arising from the left OBRU by a rotation around the rotation axis that is connected to the respective deflection line. Each rotation is shared by two OBRU's, so it is represented by two letter halves, one belonging to the outline of the one OBRU and one half longing to the outline of the other OBRU.

inward/outward orientation
Stripe paths with more levels than the basic, are concave polygons. That means that the angles in their outer border can be inward (>180º) as well as outward (<180º) oriented, viewed from inside.The alternation of inward and outward orientation, when we walk in a certain direction over the outline of such a convex polygon, is directly related to the changes in the direction of the successive rotations of the OBRU. Rotation 4 in Figure 48 is clockwise and so it results in an inward oriented angle. The other rotations are counter clockwise and result in outward oriented angles. From an analysis of the built up of the palindrome structure in that outline we can learn what's precisely the pattern in the direction switches of the successive rotations. In section 4.4 more about this.

An angle is 'outward' oriented when it's smaller than 180º and 'inward' when it's greater than 180º, reasoned from inside . Exception are the basic stripe paths represented by 0/1/1, 1/1/0 and 1/0/1[38] , which are convex polygons. We let these three further out of consideration here and focus on the concave ones. Each period consists of an unique pattern of inward /outward orientation that can be understood in terms of the layered whole of palindromes that is inherent to that period.



4.2 Zoning

Each of the units-of grid-division generates an endless range of different stripe paths, dependent on the complemented fraction s/l/m. The diversity generated by the units-of-grid-division can be divided in zones (Table II). This zonation is based on the possible combinations in variation on two properties of deflection lines : their length and the foldness of the rotation axis they are connected to. The possible length values fall in the categories s(hort), m(edium) and *ℓ*(ong) .The options in the foldness of the axes connected to those length values differ per symmetry type: 2-fold, 3-fold or 6-fold in p6 and 2-fold or 4-fold in p4 .In p3 all axes have foldness p3.
In p4 and p6 there are different combinations possible and so there are different zones. In p6 the number of combinations is greatest , because all three axis have a different foldness. In p4 two of the three axes have the same foldness, namely 4-fold. Restriction in p6 as well as p4 is that 2-fold axes cannot be combined with length category ***ℓ***. So in p6 there remain 4 possible combinations and in p4 two. In p3 there is nothing to combine because all rotation axes have the same foldness. So here is but one zone.
Understanding this zonation is important because it makes clear

zone	*ℓ*	m	s
p6: 4 zones			
I	6f	2f	3f
II	6f	3f	2f
III	3f	2f	6f
IV	3f	6f	2f
p4: 2 zones			
I	4f	4f	2f
II	4f	2f	4f
p3:1 zone			
I	3f	3f	3f

Table II: *Number of zones in p6,p4 and p3.*

[38] In p6 these represent the hexagon, the triangle and the diamond..

how wonderful subtle the transposition is of Christoffel sequences in terms of letter elements to Christoffel sequences in terms of geometric elements. In section 4.3 this shall be elucidated.

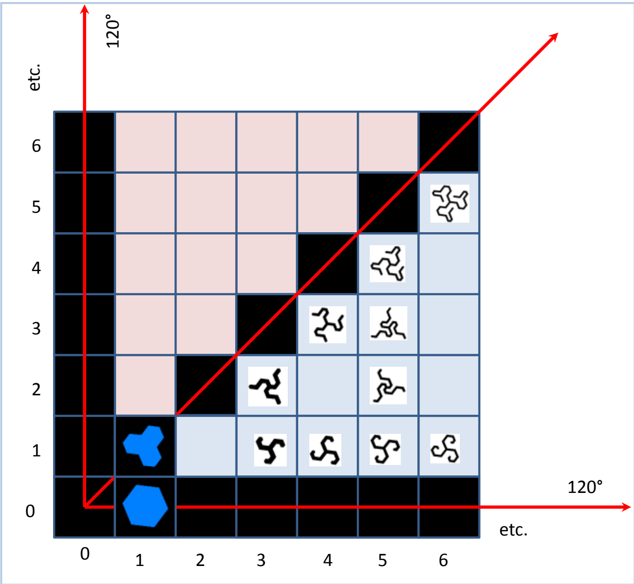
graphical presentation of the diversity

Figure 51 gives the graphical presentation of the zonation in Table II. Let us start with p4. The possible *s/m* fractions, which each represent an unique stripe path in terms of the ratio between the binary-letter-elements within it, are represented in a Cartesian Coordinate System. In section 2 (Figure 20) we already introduced this coordinate system as a manner to position the s/ℓ fractions. For s/m fractions the same principle holds. The only difference is that ℓ-values are replaced by *m*-values in the coordinate system. The vertical coordinate primary represents the possible length values of the deflection lines that are connected to the 2-fold axis of rotation and the horizontal coordinate represents the possible length values of the deflection lines that are connected to one of the two 4-fold axes. This implies that in the blue zone the vertical coordinate represents the length values in category s and the horizontal coordinate the length values in category *m*. In the pink zone the reverse applies. There are three black boundary areas in the system: a black column, a black row and a black diagonal area. In each of these, one cell is filled by a blue stripe path which can be conceived as the 'boundary marker' of the respective boundary area. In de diagonal area it is the Greek cross and in the other two boundary areas it is the square.

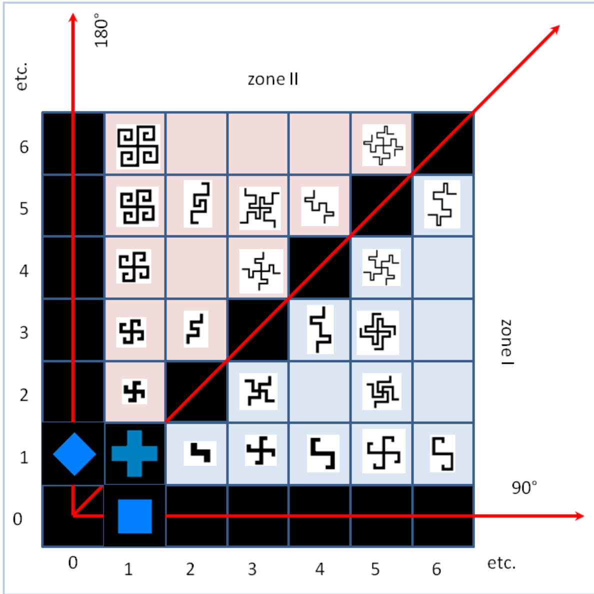
In p3 both coordinates presents length values connected to a 3-fold rotation axis. So the pink and the blue zone are identical and for that reason we have omitted the pink zone. The boundary markers here are the hexagon in the black row area and the three legged cross in the black diagonal area.

In p6 again the vertical coordinate represents the length values of the deflection lines connected to the 2-fold axis. But now in the coordinate system at the right the horizontal coordinate represents the length values of the deflection lines connected to the 3-fold axis. In that at the left the horizontal coordinate presents the length values that are coupled to the deflection lines connected to the 6-fold axis. The boundary markers here are the diamond in the black column area, the hexagon and the triangle in the black row areas and the three legged and six-legged cross in the black diagonal areas. The ordering principles in essence are the same as that in the coordinate system in Figure 32. The blue points and red point now are substituted by grid cells. The cells which substitute a blue point contain a stripe path; the cells which substitute a red point are empty.

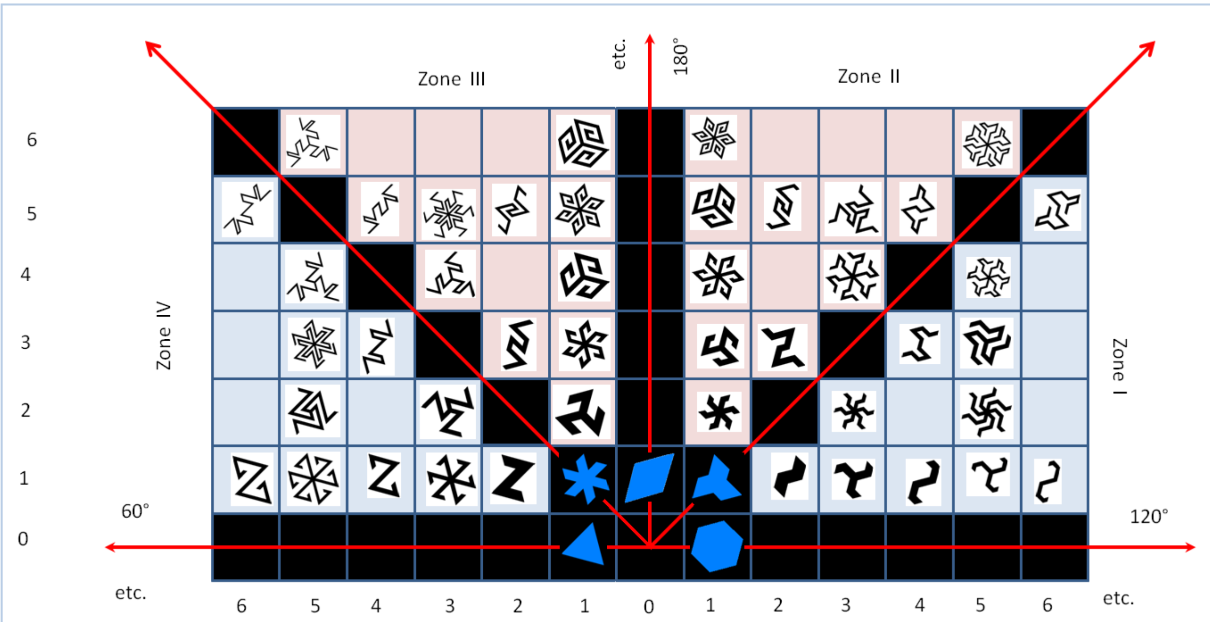
One can imagine that through these cells, like in Figure 20, blue and green lines are drawn, representing concatenations of majors which are complemented into a new stripe path by adding a minor. We have not drawn these lines in the present figure, to keep things simple. But the red lines through the border areas represent the most simple concatenations of majors, as do the coordinate and diagonal in Figure 20. It may be clear that when the number of majors in these concatenations increases, more spiral winding takes place in the stripe paths which result after adding a minor to them. So there is increasing spiral winding around the shape of the square in the cells directly adjacent to the black row and the black column, as the distance from the origin grows. And more spiral winding around the shape of the Greek cross in the cells direct adjacent to the black diagonal, as the distance to the origin grows. The same applies to all other concatenations of majors lying on imaginary blue lines through the origin. The more complex the fraction of the respective majors (in terms of the number of partial quotients) , the more complex the shape around which spiral winding takes place. In section 4.3 more about this.



p3



p4



p6

Figure 51: *Zonation in p4, p3 and p6.*

Escher once stood on the threshold.

It seems that Escher once stood on the threshold of this wonderful world. In p6, the ratio between sharp and obtuse angles reverses when you go from zone I to zone IV. Maurits Escher once was studying this reversion, which is evident from his work archives, that were opened up by Doris Schattschneider. Escher explored this reversion specifically for the ratio *s/l* = 1/4 (Figure 52).

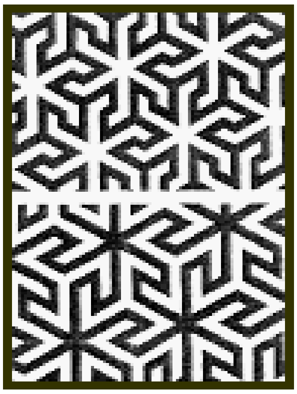


Figure 52*: Escher's study of sharp/obtuse reversion in p6 s/l = 1/4.*

Why did Escher not go further inside this mysterious part of the garden he loved so much and visited so often? May be it had to do with his preferences as an artist. Plane symmetries were the starting point for many of his creations. But most people perceive these symmetries as dry and boring, evoking associations with wallpaper. Escher, who realized this more than anyone else, invested a lot in counteracting that dullness. He filled his regular plane divisions with creepy animals that had low cuddliness, so evoking a certain excitement in that way. And he was always looking after formats with which he could give more focus and meaning to his regular plane divisions. Nice examples of this are creations like Day and Night, Reptiles coming out of the plane, Methamorphosis, Predestination, Liberation, etc.. Being so caught up to counteract the monotony that is inherent to symmetry, it may be he kept himself away from the awareness that a wonderful arithmetic world is hidden in plane symmetries, out of which beautiful abstract geometrical shapes could arise.

4.3 Layere spirals

Now the reader may be a little familiar with the phenomenon of stripe paths based on regular grid division, it's time to show how these stripe paths, being layered spirals, are the perfect geometrical conversion of all the wonderful characteristics of Christoffel sequences we presented in section 2.

As treated in section 4.1, the outer border of a stripe path is built up of f * 2 $\frac{o-PAL}{2}$'s and the inner border out of f $\frac{i-PAL}{2}$'s . The two types of palindrome halves both have the character of a layered spiral. The $\frac{o-PAL}{2}$ spirals are fully transparent in their built up while the $\frac{i-PAL}{2}$ spirals are not. The dilemma is that the $\frac{i-PAL}{2}$ spirals are the ones we are looking at when we look at a stripe path. We don't see a stripe path as a path running forth and back through the successive protrusions. We only see a number of protrusions arranged around a rotation axis. And when stripe paths become a little complex, the protrusions and the dendrites that form their inner border, look as the same. Then we are looking at the f $\frac{i-PAL}{2}$'s that share the rotation axis in the center as seat for their rpp-halves. By consequence, when we try to explain the shape of a stripe path in a way that matches with our perception, we need to explain it in terms of the $\frac{i-PAL}{2}$'s .

While I'm writing this (on Christmas eve 2015), it feels as if Elwin Christoffel, like the Ghost of Jacob Marley visiting Ebenezer Scrooge on Christmas Eve, is looking over my shoulder while whispering in my ear with malicious pleasure: My observations were not so bad after! So let this subsection, primary meant as a first global orientation, also be as a tribute to this gruff old man. It's a first view on the wonderful character of layered spirals, thereby factually looking at Christoffel's palindrome. After that, in sub-section 4.4 , we go deeper into the question how to grasp the structure of these spirals in terms of the characteristics of discrete Christoffel sequences. There the focus switches from the spirals in the inner border to the spirals in the outer border and do we leave the palindrome concept of Christoffel behind us to make way for mine.

main principle
The main principle in the built up of layered spirals is:

> Every stripe path displays in principle one or more stages in spiral winding in each of its protrusions. The shape around which that spiral winding takes place is the shape of the stripe path that arises when you delete the last value n in the series of n-values.

Figure 53 shows some examples. At the right part of the diagram a number of stripe paths is shown with, from top to bottom, an increasing number of layers. All of these display in their protrusions a substantial degree of spiral winding. At the left part of the diagram for each of these the stripe path is shown that you get when the last n-value is scored out. It is clear that the stripe paths at the left delivers the shape around which the spiral winding takes place in the protrusions of the stripe paths at the right . Let us from now on say that that more primitive stripe path at the left (of which the continued fraction has one stage less) , delivers the 'mold' for spiral winding to the less primitive stripe path at the right (of which the continued fraction has one stage more).

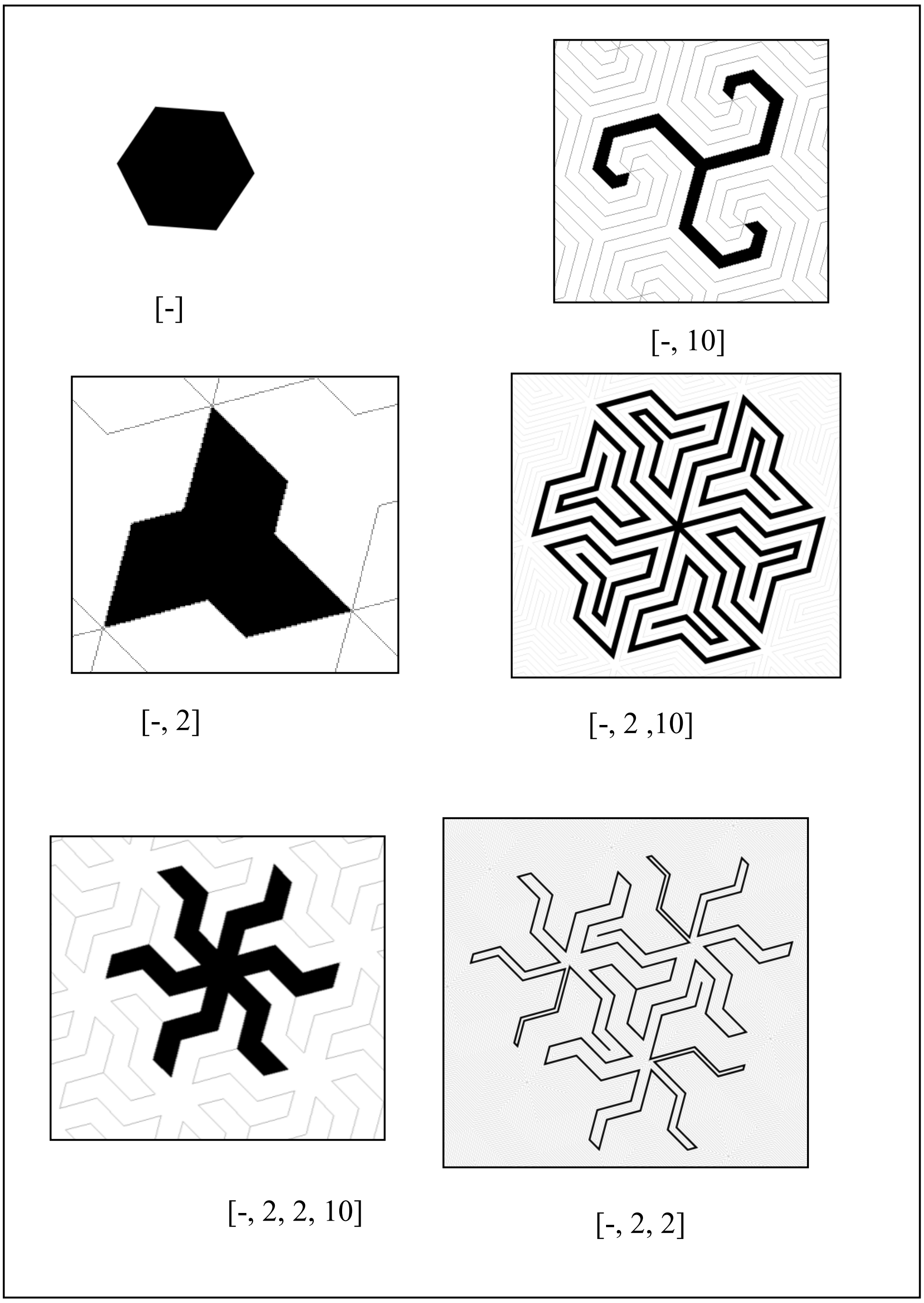


Figure: 53 *Shape of spiral winding equals the shape of the outer border of the stripe path that arises when you score out the last n-value.*

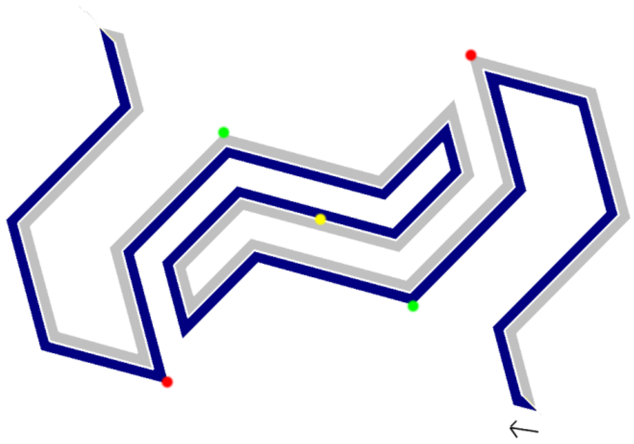


Figure 54: *Stripe path generated by s/l/m =2/12/17*

All examples above were about two adjacent levels. Now we enlarge our horizon and look at all occurring levels in the built-up of a stripe path. Let us refer to the example 5/12/17 (Figure 54) . The respective stripe path has 4 layers. Figure 55 shows the molds that are delivered by the successive layers. These molds are positioned within in the decomposition diagram of the respective Christoffel sequence, where they represent the respective periods (see Figure 18)

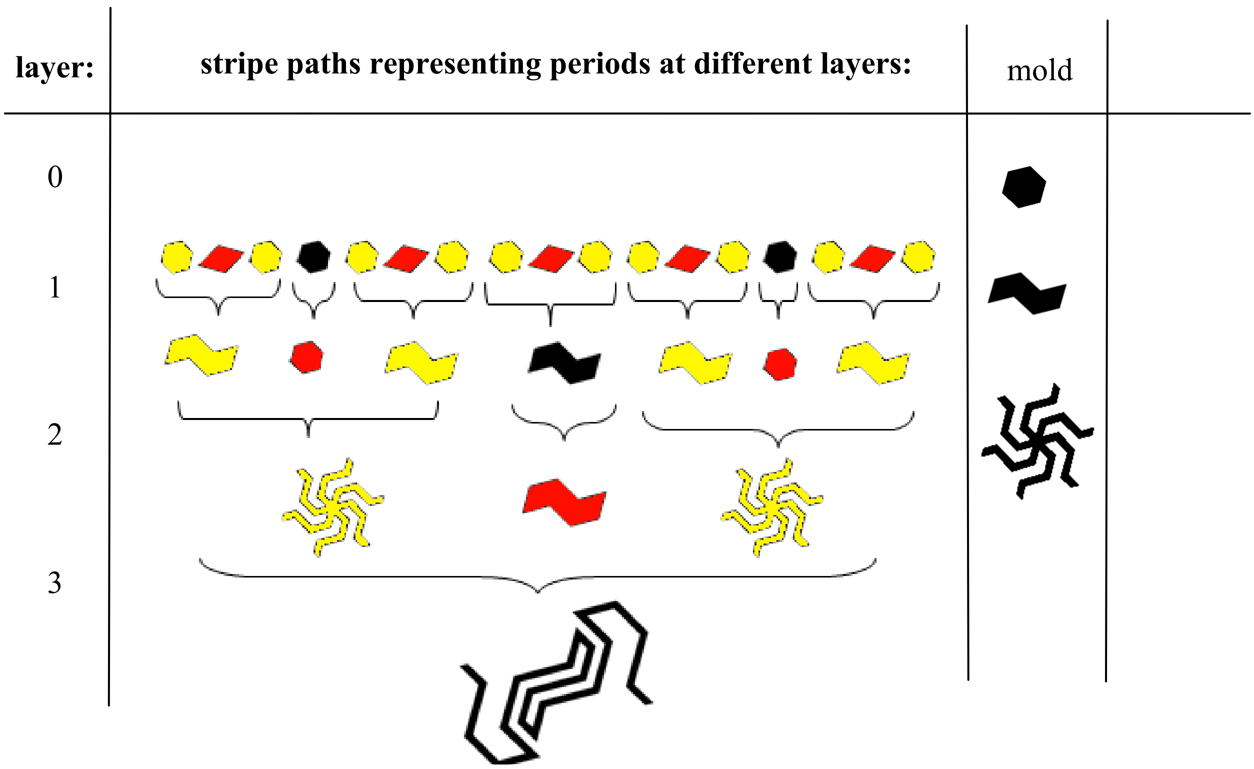


Figure 55: *Molds (black and yellow) delivered by the different layers to the following. The example s/m/l=5 /12/17.*

Figure 56 shows what happens when you enlarge the value n at one of the three levels in the stripe path. The spiral winding at that level gets more stages in spiral winding around the mold that is delivered by the preceding level.

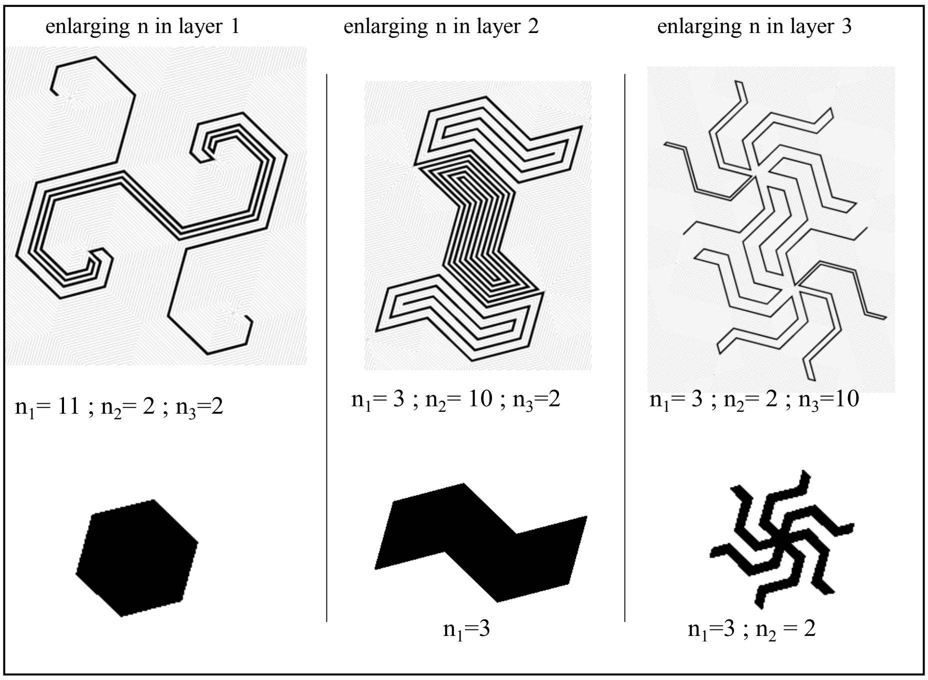


Figure 56: *Enlarging the value n at the different levels.*

Also at the levels higher that which is one higher than the mold, that increased spiral winding 'asserts' itself. Figure 57 shows this for enlarging the value n at the second level from 3 to 11. Spiral winding around the hexagon, which is the mold for spiral winding at this level, not only increases at that next level, but also at the levels there above. But there that spiral winding around the hexagon becomes included in the outline of higher order molds (zigzag, spider, etc) , that determine the course of spiral winding at higher levels than the second.

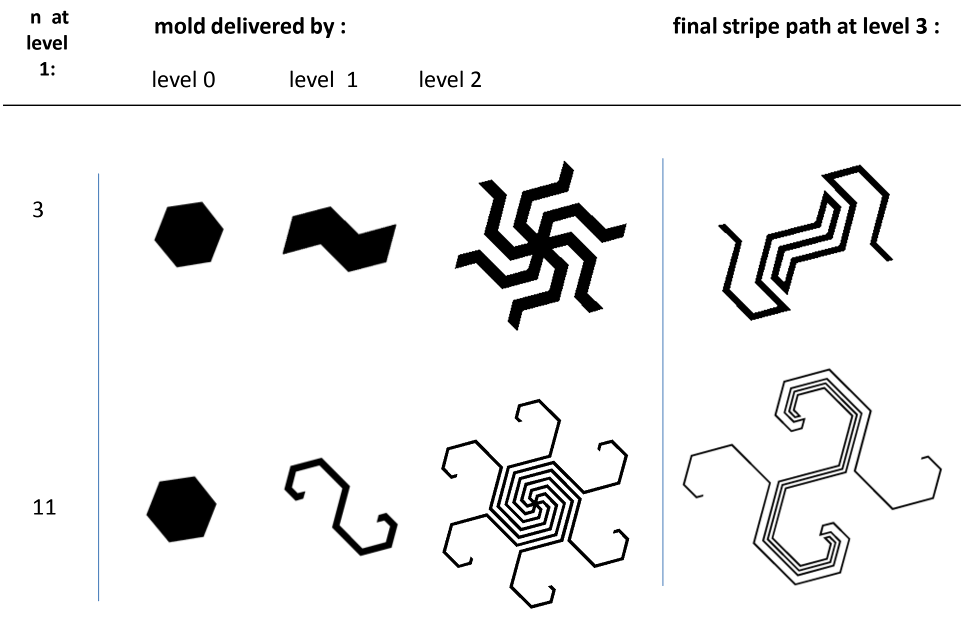


Figure 57: *Also at the levels higher than that next level, that increased spiral winding is found back, but there at lower levels in spiral winding.*

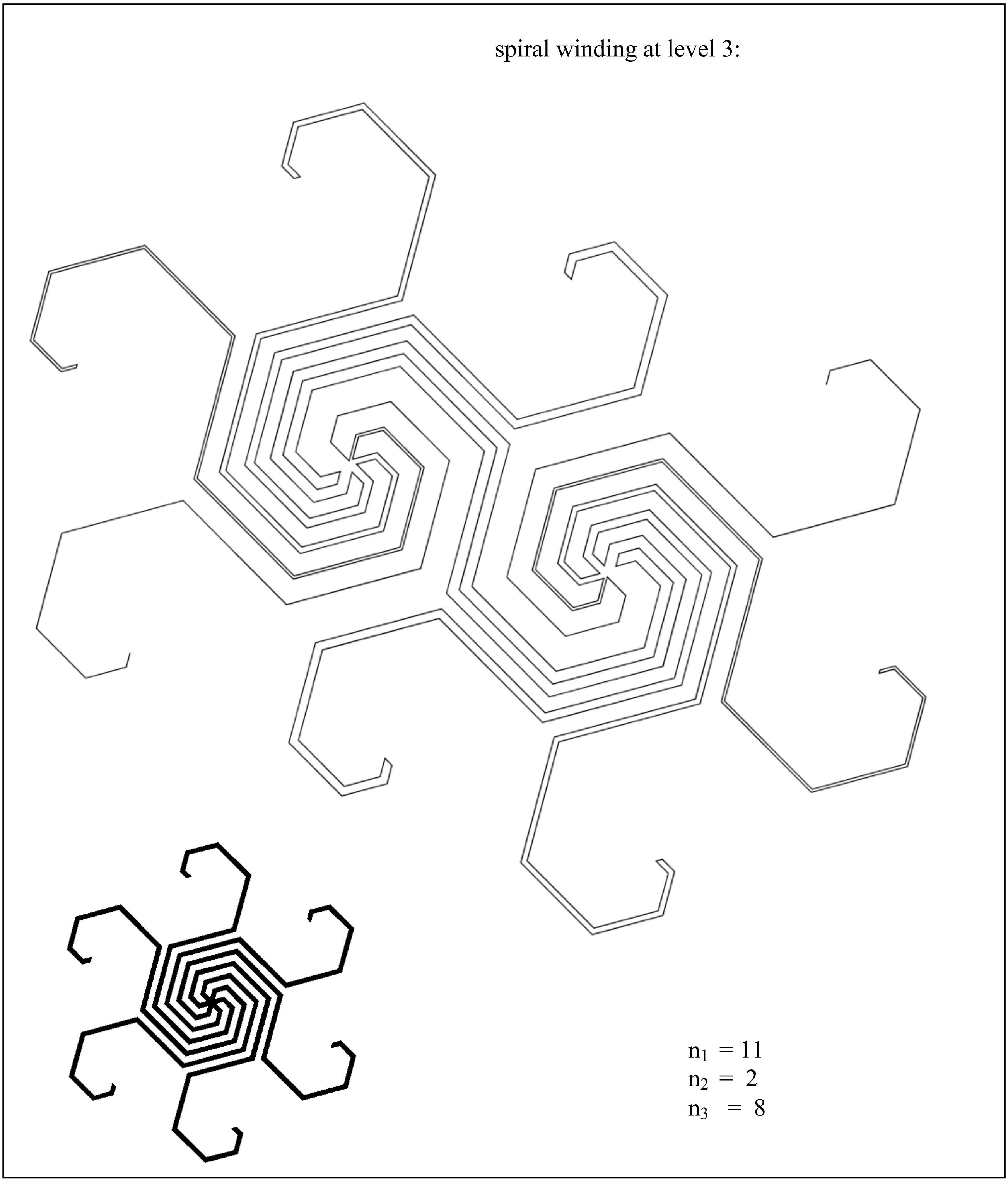


Figure 58: *Recognizing the mold delivered by level 3 in the for spiral winding at level 4.*

In Figure 57 it is not so easy to see that the mold delivered by level 3 to level 4 indeed acts as a mold for spiral winding at that last level, once we have increased the value n at level 2 from 3 to 11. But when we also enlarge the value n at level 3, (from 2 to 8 in the example,) the acting of the stripe path at level 2 as a mold for spiral winding at level 3 , immediately becomes clear (Figure 58).

multi variation
In the preceding we varied the value n at one layer and kept the values n at the other layers constant But of course we can mutually independent vary the values n at all layers. In the following I want to give some idea of this multi variation. Because the shape of the stripe path rather quickly gets a tremendous complexity when we vary the value n at more layers, let us stay by three layers , of which by definition only the two highest have a value n. Figure 59 shows first how results are when one of these two values is held at lowest level and the other is gradually increased. Then, what happens when you let increase both values n to some extent: the first to 4 and the second to 6 in the example.

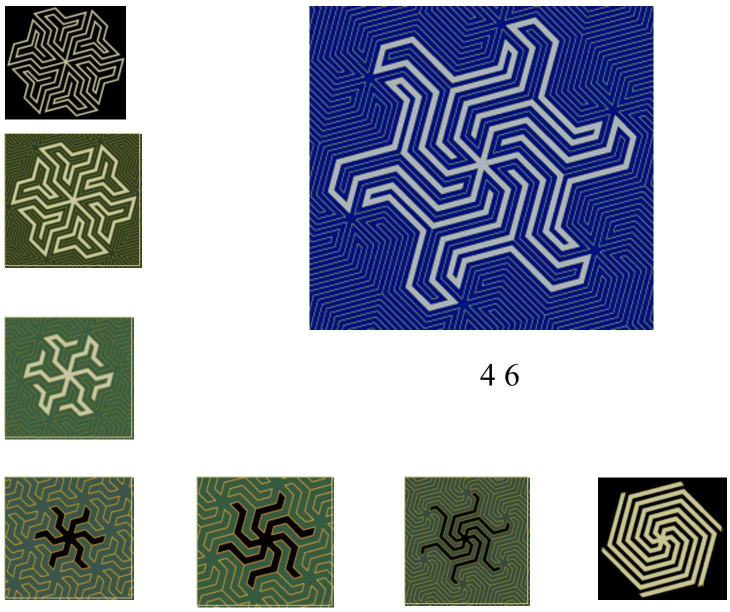


Figure 59: Multi variation in two values n in p6.

Now let's have a look at multi variation in p3. I feel a little bit pride that I can present this multi-variation, whereby the starting point is the Chinese three legged cross (Figure 60). The white cells represent the stripe paths which have three layers of which the last two have a value n. The gray cells represent the stripe paths which have two layers of which the last has a value n.. Finally the blue hexagon is the stripe paths which has but one layer an no value n connected to it.
The stripe path which has but one layer (layer 0), the hexagon, delivers its shape as mold for spiral winding for the stripe paths that have two layers (layer 0 and 1), of which the final layer has a variable n . Every value for n at that second layer results in a specific stage in spiral winding around the hexagon, which is a mold for spiral winding at the highest level of spiral winding for the stripe paths with three layers (layer 0, 1 and 2), which has as first value for n the respective value n at the second layer. The second value n specifies the stage of spiral winding around the mold.

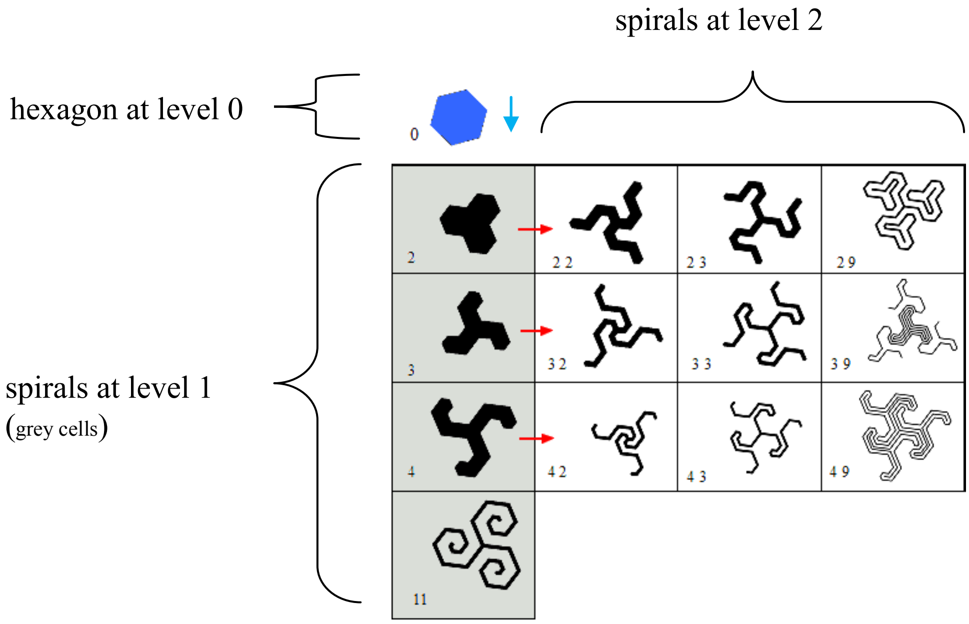


Figure 60: Multi variation in the n_i variables in p3.

Now the reader may have some impression of the wonderful phenomenon of layered spirals. The spirals directly represent the layeredness in Christoffel sequences as put forward by Henry Smith and Caroline Series. I want to finish this sub-section by making some primarily remarks about the centers of spiral winding in stripe paths. In section 4.4 more follows about the determining factors in the positioning of these centers.

coincidence of the centers of higher and lower order stripe paths

As may be clear from the just presented overview, the center of spiral winding sometimes lies in the periphery (somewhere on the outer border of the stripe path) and at other times in the center of the stripe path. Stripe paths acquire extra beauty when the center of the stripe path coincides with the center of spiraling at one or more levels. How can we manipulate values n in such way that this happens?

In the center of a stripe path lies the rotation axis that is connected to the couple of deflection lines of which the length value is even. In general centers of lower order stripe paths can coincide with centers of higher order stripe paths. In that case, of course, the lower order stripe paths only exist 'virtually', being layers in the built-up of the outline of the final layer, delivering molds for spiral winding at the respective levels. So factually we are talking about situations in which the center of a stripe path coincides with the center of the molds for spiral winding at its lower levels. The molds are specified by length values that lie in lower order convergents. In cases of coincidence of the center of the final stripe path and those of lower level molds, the couple of even length values in the final complemented fraction and those in the convergents representing lower level molds, are connected to the same type of rotation axis. But, as is easily to see in decompositions of complemented fractions (Figure 61 shows the example $s/m/\ell$ = 5/12/17 in p6 zone I) , an even number at a certain place (top, middle or bottom) in a complemented fraction, can only recur in layers that are at least two levels higher. Figure 62 shows some examples. So the mold delivered by three legged cross is a stripe path of which the center , dependent on the values n we choose, can coincide with the centers of final stripe paths that are 2, 4, etc. levels higher. We bring up the subject of coincidence of centers of higher and lower order stripe paths here so explicitly because it plays a central role in the creation of layered color mixing effects than we shall discuss at the end of this section.

With the preceding enough tribute is paid to Christoffel. Now let's look with more mathematical accuracy at the layered spirals. Thereby we focus on the outer border of the stripe paths. In the outer border the geometric appearance of Christoffel sequences is most directly and transparent.

	SL	ML	ML	SL	ML	ML	SL	ML	ML	ML	SL	ML	ML	SL	ML	ML	ML
S	1	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0
M	0	1	1	0	1	1	0	1	1	1	0	1	1	0	1	1	1
L	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

1	1	1	0	1	1	0
2	2	2	1	2	2	1
3	3	3	1	3	3	1

1	2	2
2	5	5
3	7	7

S connects 6-fold axes	5
M connects 2-fold axes	12
L connects 3-fold axes	17

Figure 61: *an even number at a certain place (top, middle or bottom) in a complemented fraction, can only recur in layers that are at least two.*

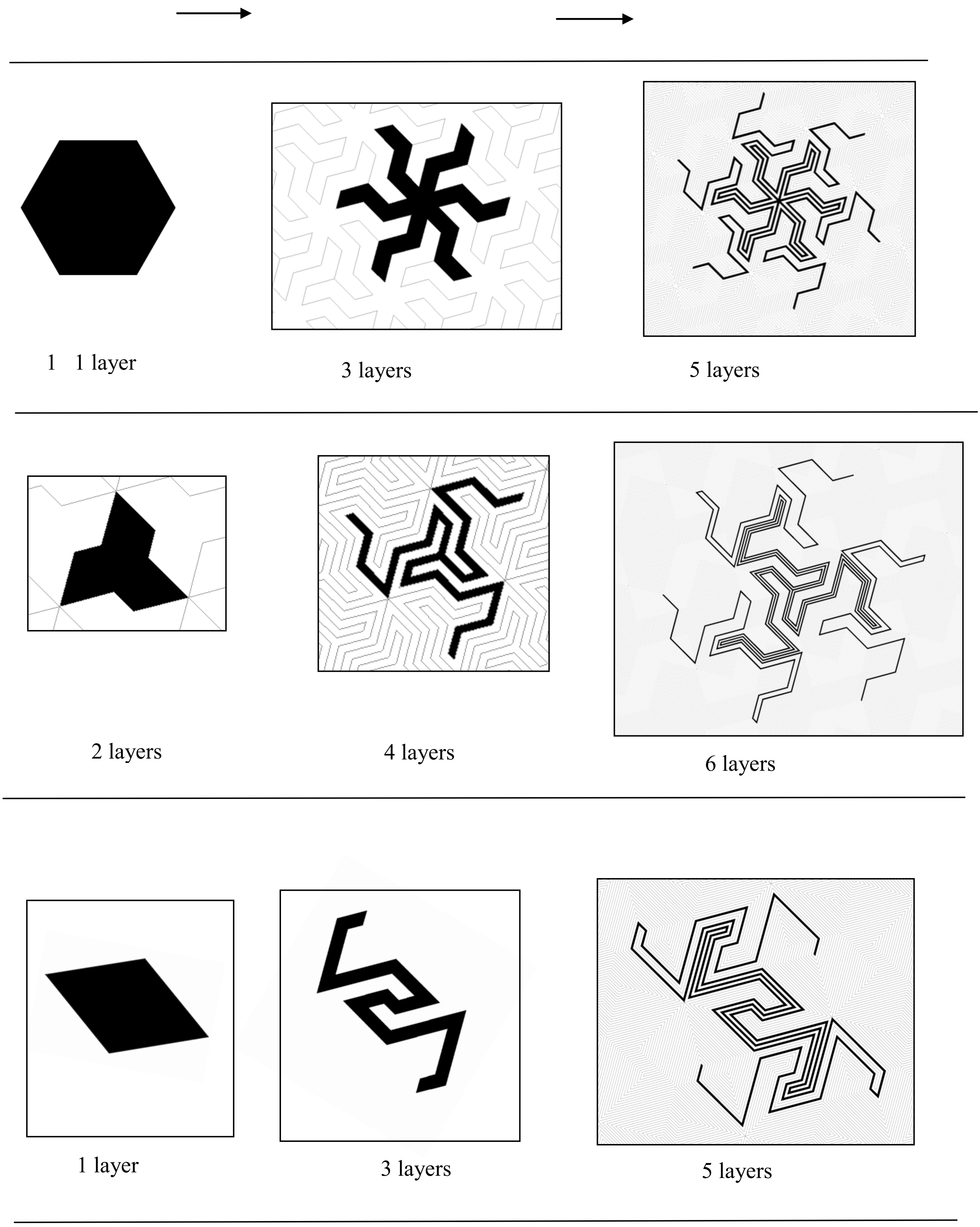


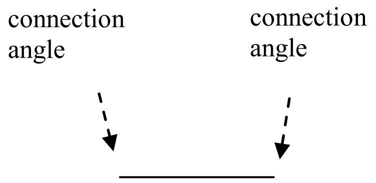
Figure 62: *Coincidence of the centers of molds at different levels with the center of the final stripe path.*

4.4 Mathematical foundation

Now we are ready to go deeper in the question how all this can be declared in terms of the characteristics of Christoffel sequences, especially in terms of their main characteristic ,namely being a layered whole of palindromes. We shall focus on the spirals in the outer border.

basic palindrome halves

As stated before, the outer border of a stripe path is built up of f o-PAL 's . (In the following I shall we omit the prefix 'o-' when talking about this type of palindrome.) Each of these consists of two $\frac{PAL}{2}$'s which are the reverse of each other. Let's first look at the palindrome halves at the level of binary-letter-elements , thus $\frac{S}{2}\,\frac{L}{2}$ and $\frac{L}{2}\,\frac{M}{2}$. These are the basic building stones in the built up of palindrome halves at higher levels, as discussed in section 2. Also in their geometrical transposition these two elements are very elemental and consist of only a line segment which at its both ends is connected to other line segements (other $\frac{PAL}{2}$'s) by angles which can have the same size or a different size:



In particular the boundary markers of the different zones, especially those lying on the horizontal and vertical coordinates, having the character of convex polygons and consisting of but one layer, are built up of these basic elements. Which connection angles occur in the pair of elements, depends on the symmetry type and in p6 it depends also on the zone types. Figure 63 presents an overview. In these boundary markers , being stripe paths with but one layer, all connecting angles are seats of rpp's and coincide with rotation axes. When there are more layers, not every connection point is a seat of an rpp.

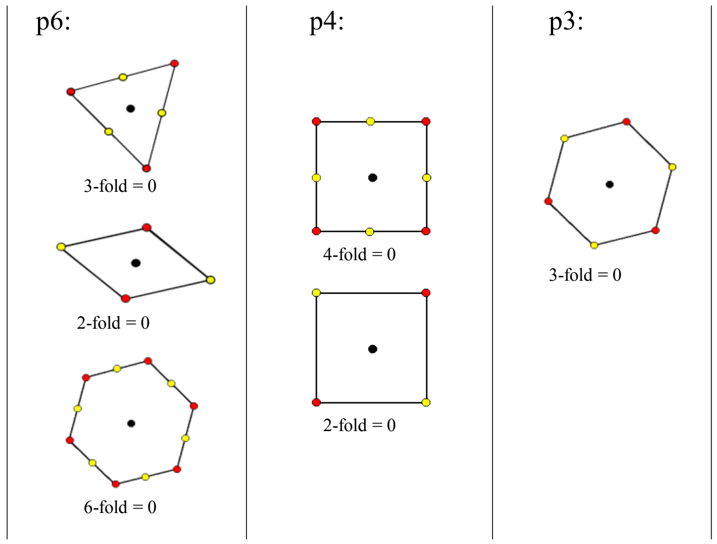


***Figure 63** : Building stones of boundary markers*

Table III shows per symmetry type the pair of basic $\frac{PAL}{2}$'s , with in p6 a further splitting based on zone type.

symmetry type	type I		type II	
	left	**right**	**left**	**Right**
p6 I and II	60°	120°	120°	180°
p6 III and IV	120°	60°	60°	180°
p4	90°	90°	90°	180°
p3	120°	120°	120°	120°

Table III: *Pair of basic $\frac{PAL}{2}$'s, per symmetry type, whereby p6 is further broken down on grounds of zone type .*

How are higher order $\frac{PAL}{2}$'s built up of these basic palindrome halves? We elucidate this for the example s/m/l = 1/3/4, in which there is but one layer above the basic layer (Figure 64). The outer border contains 2*f final $\frac{PAL}{2}$'s. So in this example there are six of them. Because there are but two levels, these six are without further aggregation built up from the two basic elements that we encounter in p6 zone I and II. Let's look more precisely at the built-up structure of one such final $\frac{PAL}{2}$, lying between a 6-fold (yellow) and a 2-fold rotation axis. Thereby we refer to the general model in Figure 22. You should consider the 2-fold axis (colored grey) as the one that lies at the top of the circle and the 6-fold axis (colored yellow) as the one that lies at the bottom of the circle. In between lie the $\frac{PAL}{2}$'s. Going from the white axis to the yellow one, you first get a primary $\frac{PAL}{2}$ at level i-1 (colored yellow), then two supplementary $\frac{PAL}{2}$'s at level i-1 (colored white) and then a $\frac{PAL}{2}$ at level i-2 (colored red) . Because n at the second level is 3, there are three $\frac{PAL}{2}$'s at level i-1 's of which one is the primary and the other two are supplementary.

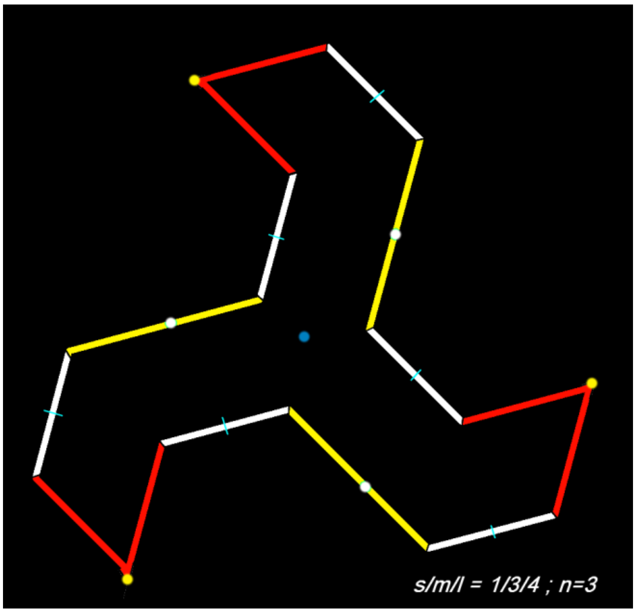


Figure 64: *Build up of the outer border of stripe path 1/4/5 out of basic* $\frac{PAL}{2}$'s.

seats rpp's

In the example, where there is only one level above the basic level, the seats for rpp's in the outline of a final $\frac{PAL}{2}$, which it does share with direct adjacent $\frac{PAL}{2}$'s at that level, coincide with the rotation axes in the outline, being of different type. There is one seat enclosed between the two primary $\frac{PAL}{2}$'s at level i-1 and one between the two $\frac{PAL}{2}$'s at level i-2. But also the basic $\frac{PAL}{2}$'s at level i-1 share seats for rpp's at their level , which are located on the connection points with other basic $\frac{PAL}{2}$'s of the same type . These seats don't coincide with rotation axis, because the basic $\frac{PAL}{2}$'s are of a lower level than the final level. Further on , where we discuss the building-up of palindromes with more than one level above the basic one, more about this.

inward/outward orientation of the angles
In the example in Figure 64 it is clear that there lie inward oriented angles at the one side of each of the two types of rpp's (coinciding with a rotation axes in the outline) and outward oriented angles at the other side. The angles that coincide with the rpp's all are outward oriented. Further on, where we treat the inward/outward orientation in stripe paths that have more than one layers above the basic, the rules behind inward/outward orientation are elaborated in more detail. For these stripe paths the rules are more complex than this simple example suggests.

higher order palindrome halves
Each palindrome that has a higher level than the basic level is built up of two halves which have the character of a spiral (Figure 65).

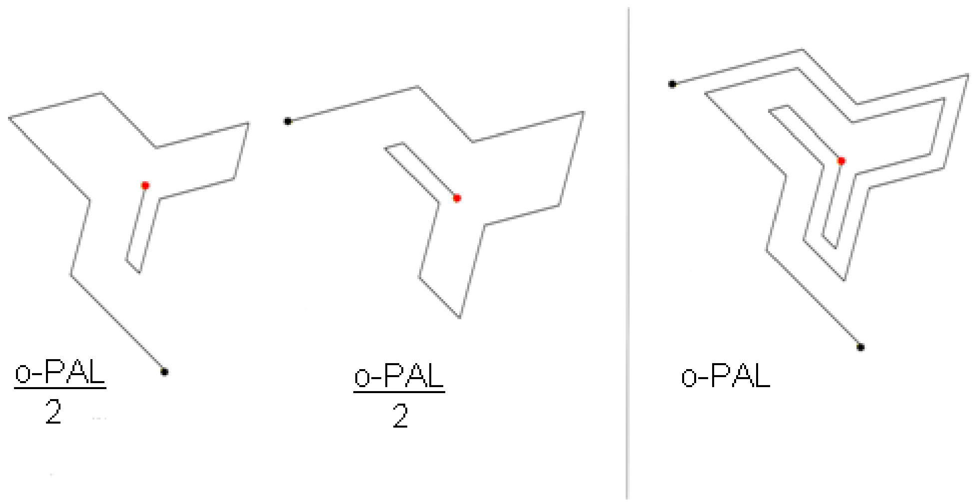
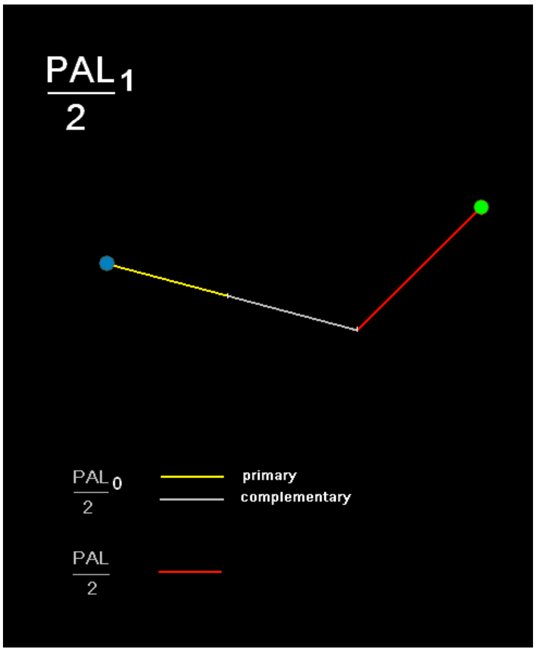
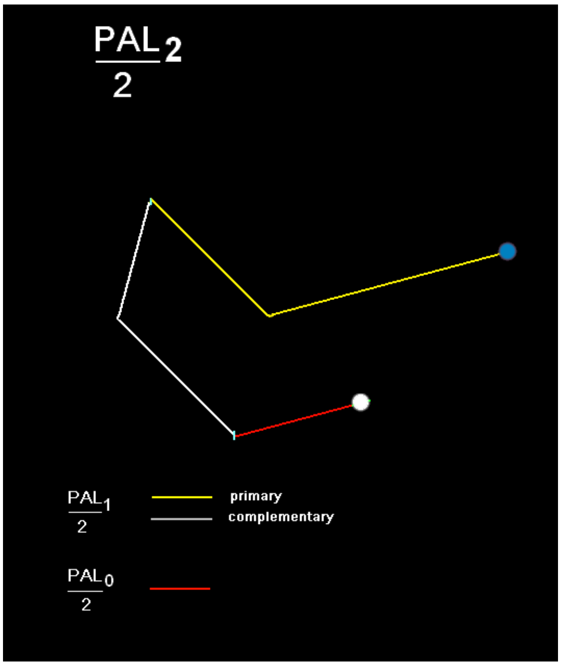


Figure 65: *Each PAL in the outer border of a stripe path consists of two* $\frac{PAL}{2}$ *'s which are the reverse of each other. Each of these two* $\frac{PAL}{2}$ *'s has the character of a spiral*

These spirals have a layered built up. Speaking in terms of palindromes: The $\frac{PAL}{2}$'s at level i, being spirals , are built-up of $\frac{PAL}{2}$'s at level i-1, which represent the successive stages in the spiral winding in the spirals at level i. Each of these $\frac{PAL}{2}$'s at level i-i itself also is a spiral, now built up of $\frac{PAL}{2}$'s at level i-2, that again represent the successive stages in spiral winding in the spirals at level i-1, etc.

n the following a closer look at this building-up of spiral winding. Let's forget for a moment the spiral like character of $\frac{PAL}{2}$'s and talk in general terms about the layered built up of the geometrical version of $\frac{PAL}{2}$'s, as they manifests themselves in the outer border of stripe paths . In the preceding (Figure 64) we gave already a preview , elucidating the built up of the geometrical version of $\frac{PAL}{2}$'s which consist of two layers: layer 0 and layer 1. Now we present the general idea, also applicable to more complex stripe paths. We again use the example 5/12/17 to show how the layered built up of the geometrical version of $\frac{PAL}{2}$'s can be grasped in terms that the model presented in Figure 22. Figure 66 shows this built up for *s/m/l* = 5/12/17.





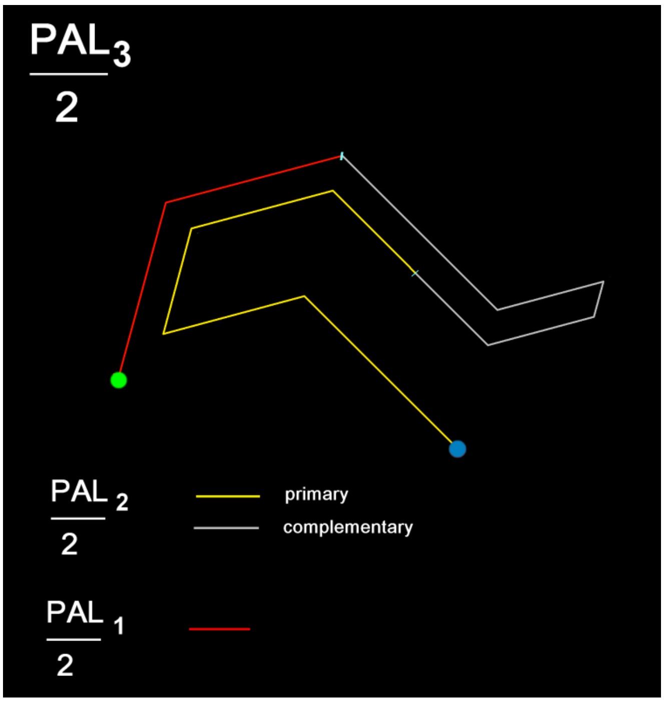
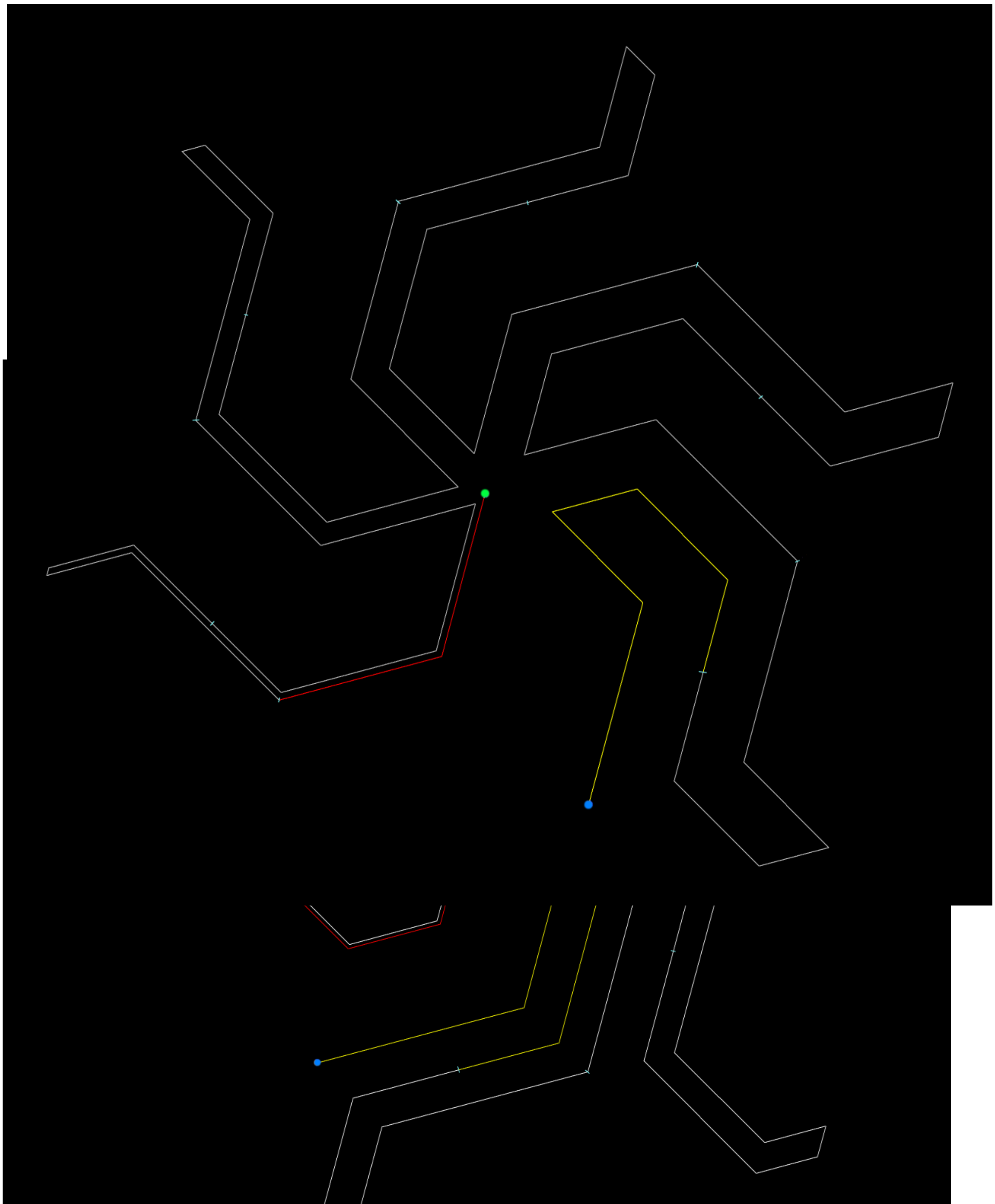


Figure 66: *Built-up* $\frac{PAL}{2}$ *s/m/l = 5/12/17. At the first,second and third layer the value of n is 2, which means that at all these layers there is but 1 stage in spiral winding.*

Each $\frac{PAL3}{2}$ is built up of two $\frac{PAL2}{2}$'s , one primary (yellow) and one supplementary (gray), that are complemented by a $\frac{PAL1}{2}$ (red). Each of the $\frac{PAL2}{2}$'s within that $\frac{PAL3}{2}$ in itself is built-up of two $\frac{PAL1}{2}$'s, one primary (yellow) and one supplementary (grey), complemented by a $\frac{PAL0}{2}$ (red). And each $\frac{PAL1}{2}$ within a $\frac{PAL2}{2}$ in itself is built-up of two $\frac{PAL0}{2}$'s , one primary (yellow) and one supplementary (grey), complemented by a $\frac{PAL}{2}$ (red).

spiral characteristics of the $\frac{PAL}{2}$'s

In $\frac{PAL3}{2}$ in Figure 66 the value n is 2 in all three layers which have n values. So there is but one stage in spiral winding at each layer . Consequently, when we look at that structure, it is not so clear that we have to do with a layered spiral . When we increase the value n it immediately becomes clear. Figure 67a shows the increase an spiral winding at the highest level, level 4, when we increasing the value n for that layer from 2 to 10. More supplementary $\frac{PAL}{2}$'s are added to the primary one in layer i-1 . Every two extra $\frac{PAL}{2}$'s result in an extra stage in spiral winding. So there are 4 extra stages in the spiral winding at level 4 in this figure. But of course we can at the same time increase n-values at other levels. In the example in Figure 67b, except increasing n_3 from 2 to 10, also n_1 is increased from 2 to 4, which result in more spiral winding around the mold of the hexagon at level 2 .While the center of spiral winding at level 4 lies in the center of the spiral, those at level 2 lie in the periphery.



b)

Figure 67:
a) When n at layer 3 is increased from 2 to 10, the number of $\frac{PAL}{2}$ *'s is increased from 2 to 10. By consequence the number of stages in spiral winding at this level is increased from 1 to 5.*
b) When also n_1 *is increased from 2 to 4, this result in more spiral winding around the mold of the hexagon at level 2 .*

Centers of spiral winding

In every layered spiral there are multiple centers of spiral winding. Some are 'active' at only one layer and some at more layers. In geometrical perspective these centers all coincide with rotation axes that lie around the outline of the spiral. Within the letter sequence these centers are represented by the r.p.p's that lie in the center of the red zones that we can discriminated at the different levels in the decomposition diagram. Figure 23 illustrated the example *s/m/l* = 5/12/17. Figure 68 illustrates how the respective r.p.p's can be traced by means of this decomposition diagram.

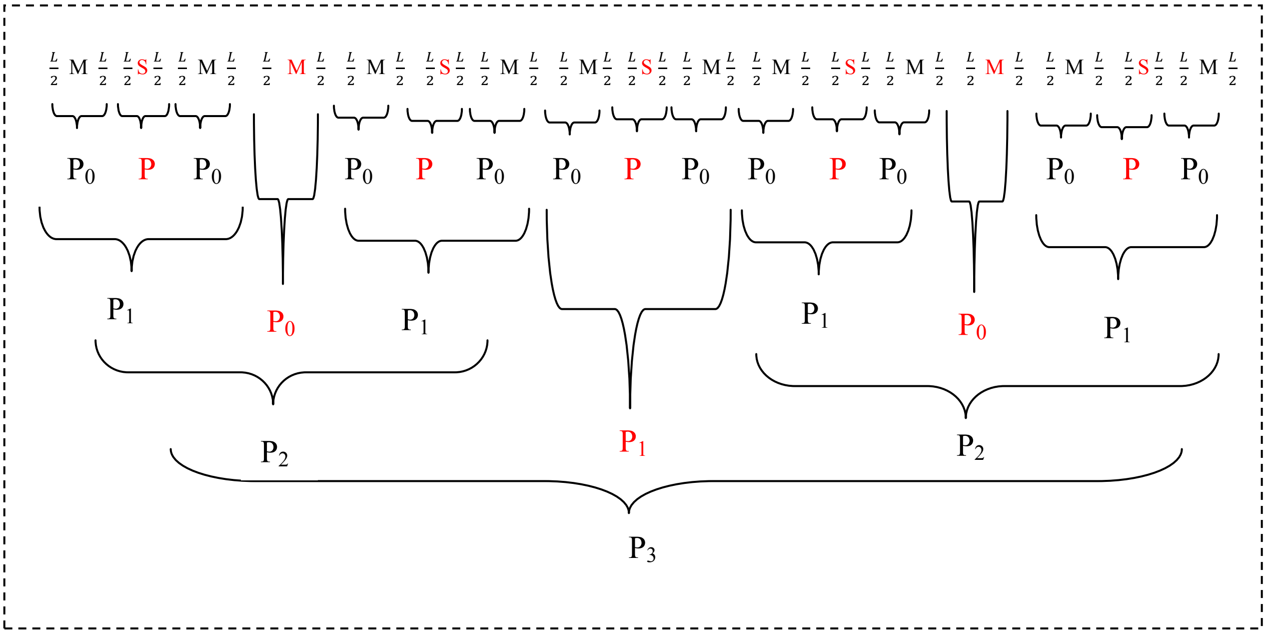


Figure 68: Location of the r.p.p's that represent centers of spiral winding in the decomposition diagram of example 5/12/17. The respective r.p.p's are colored red.

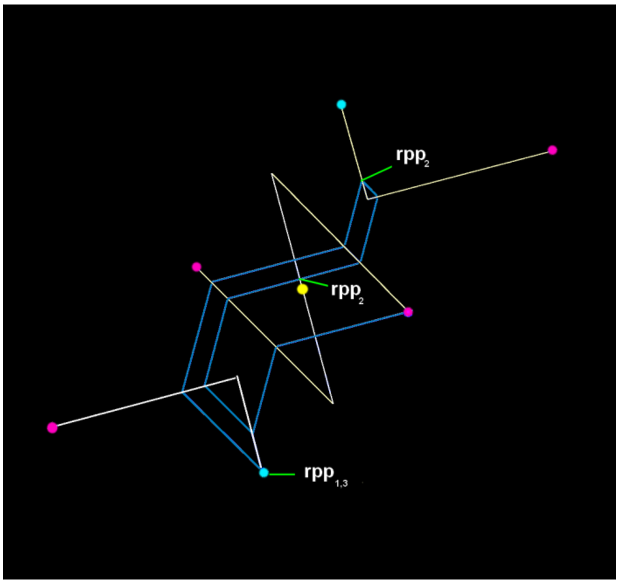


Figure 69: Rotation axes as the real centers of spiraling.

There always lies a rotation axis at each of the ends of a spiral. Only one of these is center of spiraling, namely the one that lies at the end of the red zone (see Figure 66). Let's take the example *s/m/l* = 5/12/17 presented in Figure 69. The blue line presents the spiral ; the white lines are the deflection lines. At each end of the spiral lies a rotation axis: a green (6-fold) at the one end and a purple axis (3-fold) at the other. The green is center of spiraling. When there are more layers in the spiral, as in this example, the axis is center of spiraling at more than one level. And by consequence it coincides with a letter which is seat for an rpp that's center of spiraling at the same levels. In the example the rotation axis and the rpp that coincides with it are active at level 1 and 3. None of the other rpp's that are center of spiraling coincide with a rotation axis. They all lie on a deflection line that is connected to a rotation axis that is the real center of spiraling. In the example there are two other rpp's active: of which one at level 2 and the other at level 1. Both lie on a deflection lines that is connected to a rotation axis that is active as center of spiraling.

In the example there are all together 3 centers of spiral winding active of which one at two levels, as shown in Figure 68 and 69. But far more centers can be active, when the value n[39] increases. Figure 70 shows the geometrical spreading of the three centers that are active in the example and the centers that can become active in addition to those three, when the valu n increases[40].

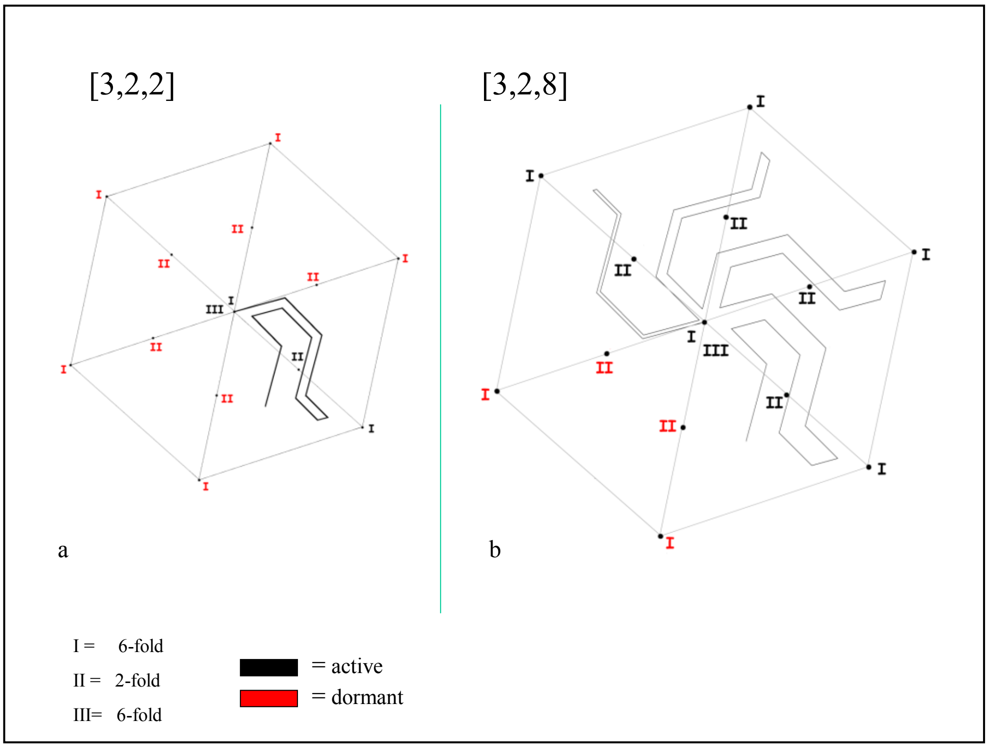


Figure 70: *Geometrical spreading of the centers of spiral winding for s/m/l= 5/12/17.*

They all lie on a hexagon which has a 6-fold rotation axis in its center. This axis is center of spiral winding at level 1 as well as level 3 The 2-fold rotation axes, which are center of spiral winding at level 2 lie in the middle of the 'spokes' of the hexagon. The remaining centers of spiral winding, all at level 1, lie in the angles of the hexagon. Most of the specified centers are not 'yet' active as center ('dormant'), because the scope of the final spiral winding is small. When the value of n_3 is enlarged and that scope broadens, more centers become active (Figure b). The spreading scheme of the centers of spiral winding may take various geometric forms, dependent on the distribution of odd and even in the numerators and denominators within the series of convergents in the continued fraction. It can take the form of a hexagon, a triangle, a diamond, a hexagon elongated in two sides, concentric hexagons, etc. The more layers in the continued fraction, the greater the number of centers included in the scheme and the greater the number of levels in spiral winding in which the centers near the midpoint are involved.

[39] In this example the value n of level 3.
[40] In all the possible spirals for which the format of the series of partial quotients is [o,e,e), where 'e' are even numbers and 'o' uneven numbers.

4.5 The mystery of concave polygons resolved.

Having cleared up so much concerning my stripe paths, there still remained something in their structure that was unfathomable, even after the uncovering of the internal structure of the o-PAL. That became painfully apparent when, last year, I tried to discuss my ideas with some mathematicians in the area of Combinatorics On Words. I wrote them about my wonderful stripe paths based on Christoffel sequences, thereby specific referring to the picture in Figure 71. It shows how these stripe paths, meandering through the plane, thereby traversing longer and shorter deflection lines, generate a Christoffel sequence. "We agree that the stripe path generates a Christoffel sequence", was their answer, " but we don't see how the shape of the figure is precisely related to that sequence." They referred to the fact that my stripe paths are complex concave polygons of which the outline is built up of angles that alternately are inward and outward oriented when one walks along that outline in one of the two directions (see Figure 72). That alternation pattern doesn't simply match with the alternation pattern of L and S. But it does match with the model of the layered built-up of the o-PAL in Figure 22. At every level in the built-up of a final PAL there are two PAL halves which are the reverse of each other in terms of inward and outward orientation, when one walks along the outline in one of the two directions. But it is not quite that simple. The angles that coincides with the r.p's of those PALS's can only be inward or outward oriented, which complicates the matter. To get fully grip on the structure in terms of inward/outward orientation of the successive angles, a slight adjustment is needed in the model. But taking about adjustments in my model for the layered built up of the o-PALwas 'a bridge to far' for those mathematicians:

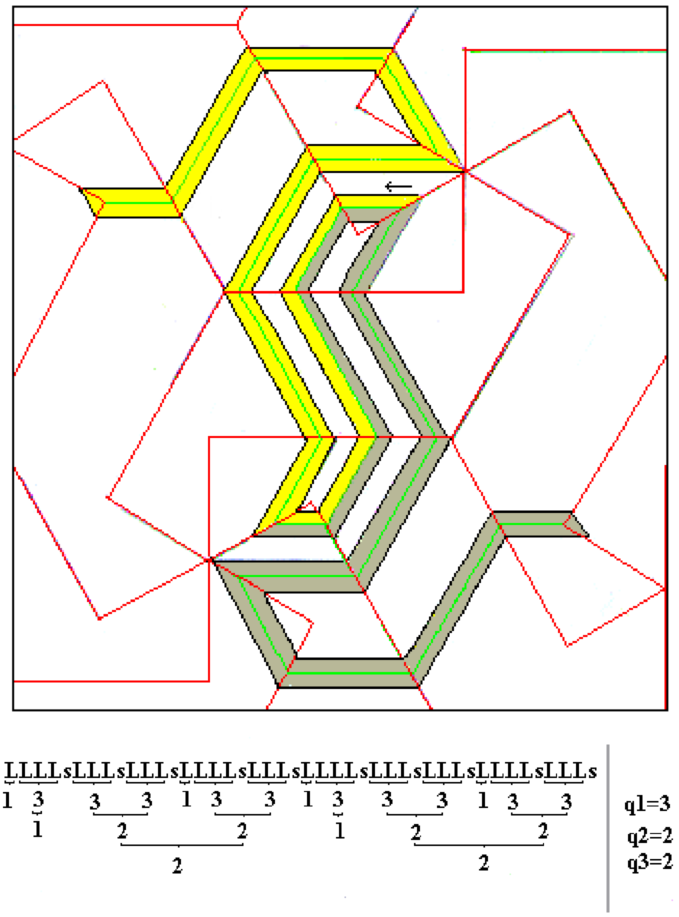


Figure 71: *Meandering through the plane and by that alternately traversing longer and shorter deflection lines, the stripe path generates a Christoffel sequence.*

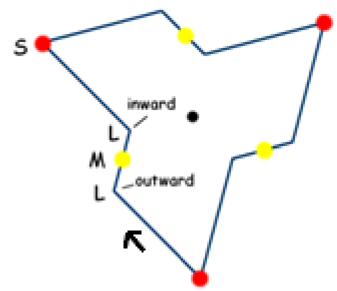


Figure 72: *Successive angles are alternately outward and inward oriented when you walk along the outline of s stripepath in one of the two directions.*

- those guys don't at all recognize the o-PAL ;
- they have little attention for the layered character of Christoffel sequences as once revealed by Henry Smith an later by Caroline Series;
- worse of all, they don't discriminate rpp's in palindromes, what their great inpirer Elwin Christoffel suggested them to do .

I advised them to recognize my o-PAL beside Christoffel's i-PAL and pointed them to the importance if the r.p.p , writing them that in my view that the rpp is one of the most fascinating subjects of discrete mathematics. After this a little megalomaniacal effusion from my side, there was only silence from their 's, as always is the case when I try to communicate about my all scorching passion with the big humanworld, full of competent and settled people who have to do better things then listen to the stirrings of soul of a weirdy like me, having no other mission in life than exploring structures of perfect order. So they never shall understand my stripe paths, in spite of the hundreds of pages they have devoted to the formulation and prove of a plurality of theorems regarding Christoffel sequences.

little adaption to the model

Anexpose about necessary adjustment in my model need to begin with some remarks about a notation system with regard to patterns in inward and outward orientation.

symbols for inward/outward orientation

The separate letters in a sequence represent angles that can be inward and outward oriented. We indicate an angle as outward oriented by placing a '+' sign before the respective letter and as inward by placing a '-' sign before the respective letter. To indicate that a whole sequence of letters is the counterpart of another sequence of letters, we use the symbols :

$$\overbrace{\ldots\ldots\ldots}^{+} \qquad \overbrace{\ldots\ldots\ldots}^{+}$$

What is indicated with '-' and what is indicated by ' +' at this aggregated level, is arbitrary. So let us use '+' when the sum of all individual + is greater than zero.

model inward/outward orientation

To make my model better suitable to understand the patterns of inward and outward orientation in the outline of my stripe path, some adaption of the model is needed. Figure 73 shows this adapted model. It makes clear that the two rpp's in the final inter-PAL, that lie opposite to each other on the mirror axis, always are outward oriented. In between, all elements in the sequence are the counterpart of each other as to inward/outward orientation, starting from the mirror axis. When an element is

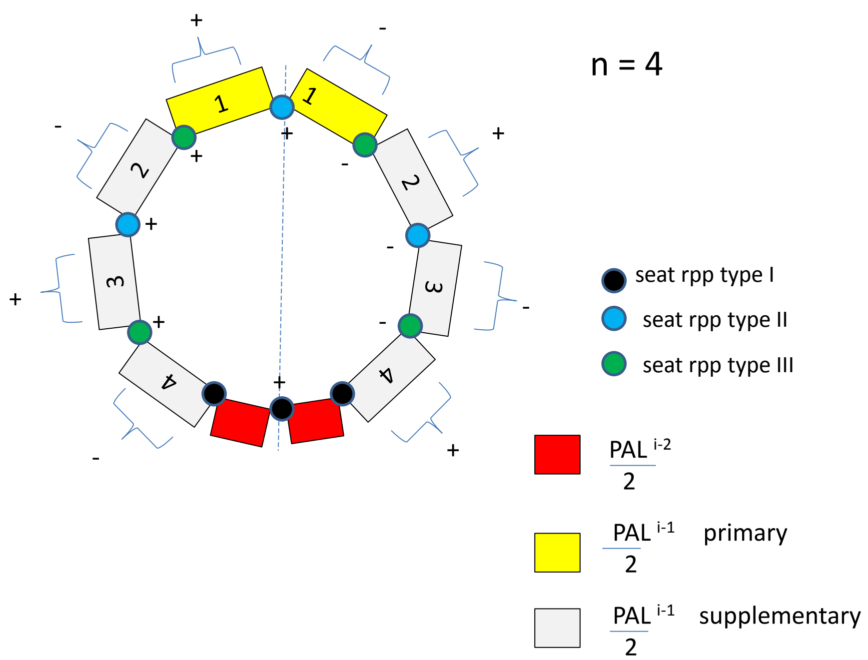


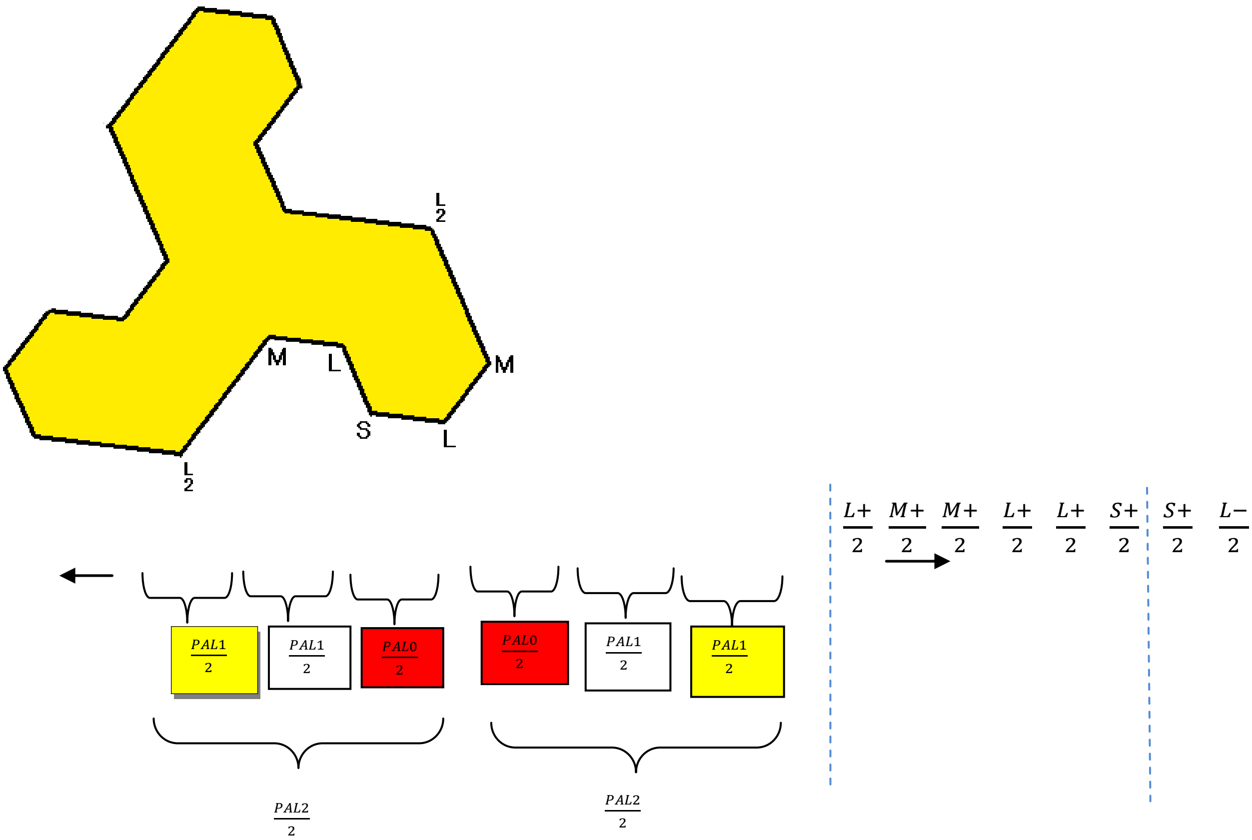
***Figure 73:** Model for inward/outward orientation of the angles in a palindrome.*

inward oriented in the left column, it is outward oriented in the right column At first glance the principle looks simple but in on closer inspection it is rather complex:

- As stated, the rpp's that lie on the mirror axis , representing the final level i, are always outward oriented.
- The $\frac{PAL}{2}$'s at level i-1, at the left, while the same goes for these $\frac{PAL}{2}$'s at the right, on the level of sequences as a whole are alternately + and - oriented , except the shared rpp's that lie on the connection points between them , that all have the same orientation.
- As such the patterns of inward/outward orientation at the left and at the right of the mirror axis are the counterpart of each other, with exception of the two rpp's at the top and at the bottom, which according to the same principles are similar oriented , and in this case, because they are also rpp's at the final level, both outward directed).

.

Application of of the adapted model is the same in all three types of regular grid division. But in p4 and p6 there are angles of 180º, which is an extra complication. So let 's focus on regular grid division in p3. The principles that we discuss there, can simply be transferred on p4 and p6[41]. We start with the example in which the second layer is the final layer (Figure74). The example is *s/m/l* = 1/2/3 which implies that n=2.

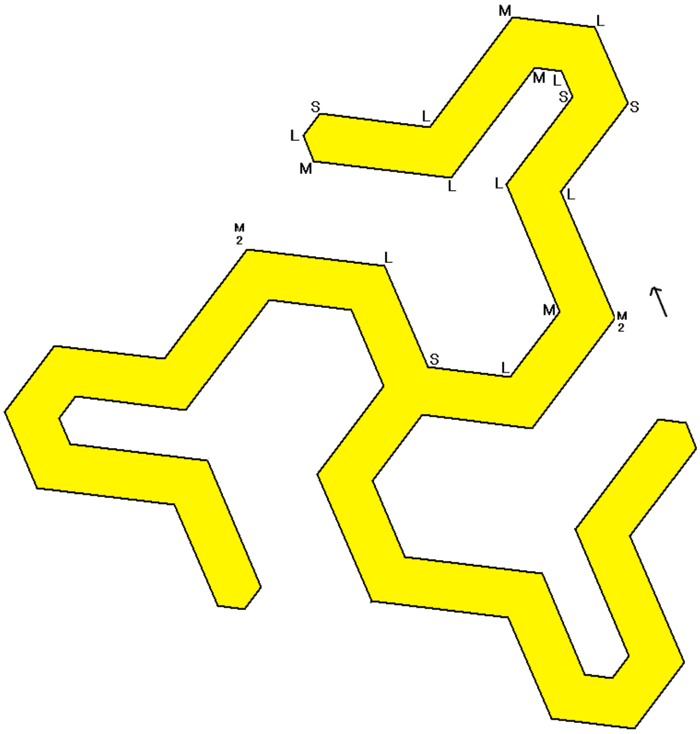


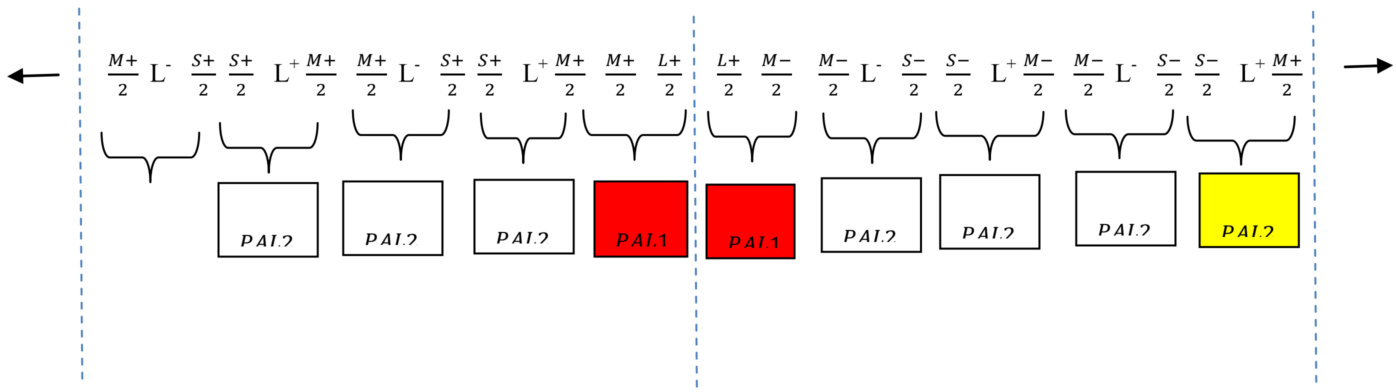
41 In p4 and p6 we qualify angles of180º as inward or outward oriented, dependent on their position within the sequence.

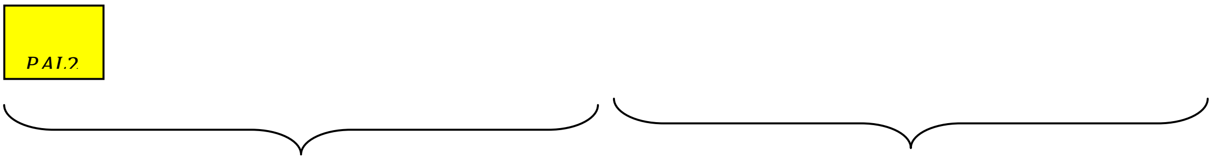
Looking at the sequence of letters representing this stripe path and referring to the model above, the mirror axes through the letter sequence, going through a pair of $\frac{L}{2}$'s and a pair of $\frac{S}{2}$'s, matches with the mirror axis in the model. The respective $\frac{L}{2}$'s lie at the top of the mirror axis in the model and the respective $\frac{S}{2}$'s at the bottom. They are the seats of the final rpp's. In between lie letters that are each other's counterpart, considered from the mirror axis. Starting from S, going contra clockwise through the outer border (from the perspective of the center), the respective angles are inward oriented and going counter clockwise trough the outer border they are outward oriented.

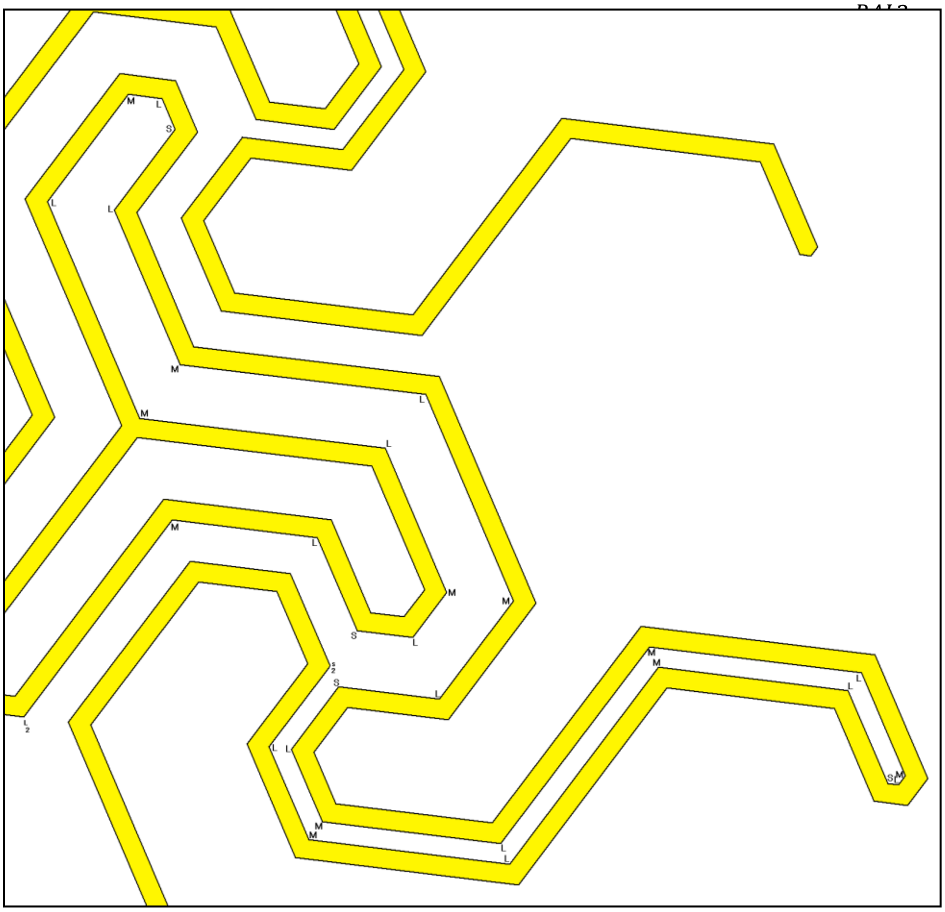
Because n=2 there are two $\frac{PAL1}{2}$'s in the built up of each $\frac{PAL2}{2}$, one primary (yellow) and one secondary (gray), complemented by a $\frac{PAL0}{2}$ (red). It is clear that the two primary $\frac{\text{PAL}}{2}$'s at level 1 are not fully each other's counterpart because these share a final rpp. Final rpp's are always outward oriented. The same applies to the two $\frac{\text{PAL}}{2}$'s at level 0.

As to the remaining model complexity just described, this example doesn't manifest that because the $\frac{\text{PAL}}{2}$'s are all $\frac{\text{basic−PAL}}{2}$'s and these consist only of connection points. There is no internal structure between the connection points. The connection points all the same orientation at the one side of the model and the contrary orientation at the other side of the model. There is no alternation here between + and - orientation on the level of $\frac{\text{PAL}}{2}$'s as a whole here. That's different in the next example, where the stripe path has one level more (Figure 75).

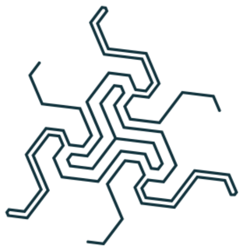








In this example it's clear that in between the connection points



between the $\frac{PAL}{2}$'s at level 2, the sequences 'as a whole' are alternately '+' an '-' oriented, starting left from the mirror axis with '+' and right from the mirror axis with '-'. But in this example these sequences consists of only one letter.

Let us look at an example in which there are more letters in these ' in between' sequences. Therefore we need to go at a stripe path with 4 levels. We choose 9/22/31 (Figure 76). Because of the length of the sequence, we focus on only one of the two final $\frac{PAL}{2}$'s , which in this example are $\frac{PAL}{2}$'s at level 4.

Figure 75: *Two levels in inward-outward orientation . The example is s/l/m= 4/5/9.*

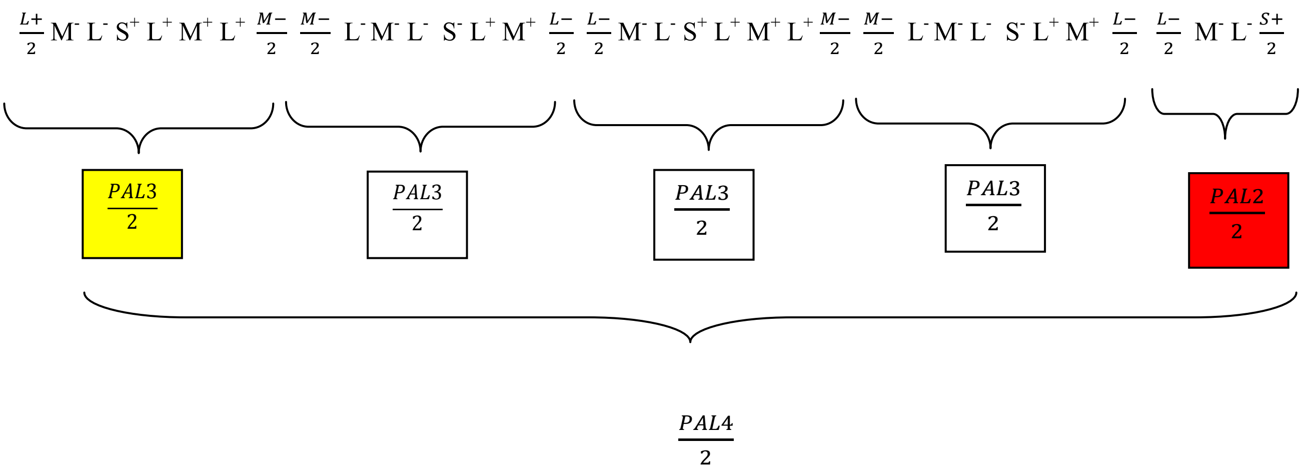


Figure 76: *Three levels in inward-outward orientation.The example is 9/22/31.*

The final $\frac{PAL}{2}$, being of level 4, is built up of 4 $\frac{PAL}{2}$'s at level 3 which, as in the preceding example, alternate in orientation. But in this more complex example it becomes also clear that equal positioned letters within the $\frac{PAL}{2}$'s at level 3 , when these PAL's as a whole are different oriented, in their orientation are each other's counterpart. With exception of the letters that lie on the connection points between these PAL's at level 3. These are all the same oriented per $\frac{PAL}{2}$ at level 4. Except the letter halves which coincide with the final rpp's and are connection points between the two $\frac{PAL}{2}$'s at level 4 and lie on the morror axis in the model. These are always positive oriented.

4.6 Creating beauty

As may have become evident from the preceding, stripe paths are a very effective showcase for displaying the beauty, elegance and magic of Christoffel sequences. All sorts of wonderful figures can be generated by manipulating the values n in the underlying Christoffel sequences. Until now we were focussed on single figures. But also the patterns that are built up of these figures can be made more of less intriguing, dependent on how we choose values n. Figure 77 is an example. The respective fraction is 21/34/55. The partial quotients of *s/m* are 1 1 1 1 1 1 2. By consequence the pattern contains a huge number of levels in spiral winding of which Figure 78 presents the molds. Because all partial quotients are very low, the degree of spiral winding around all these molds remains very small with as consequence that the respective levels in spiralwinding are only faintly shining through in the patter. It makes the pattern mysterious ,unfathomable.

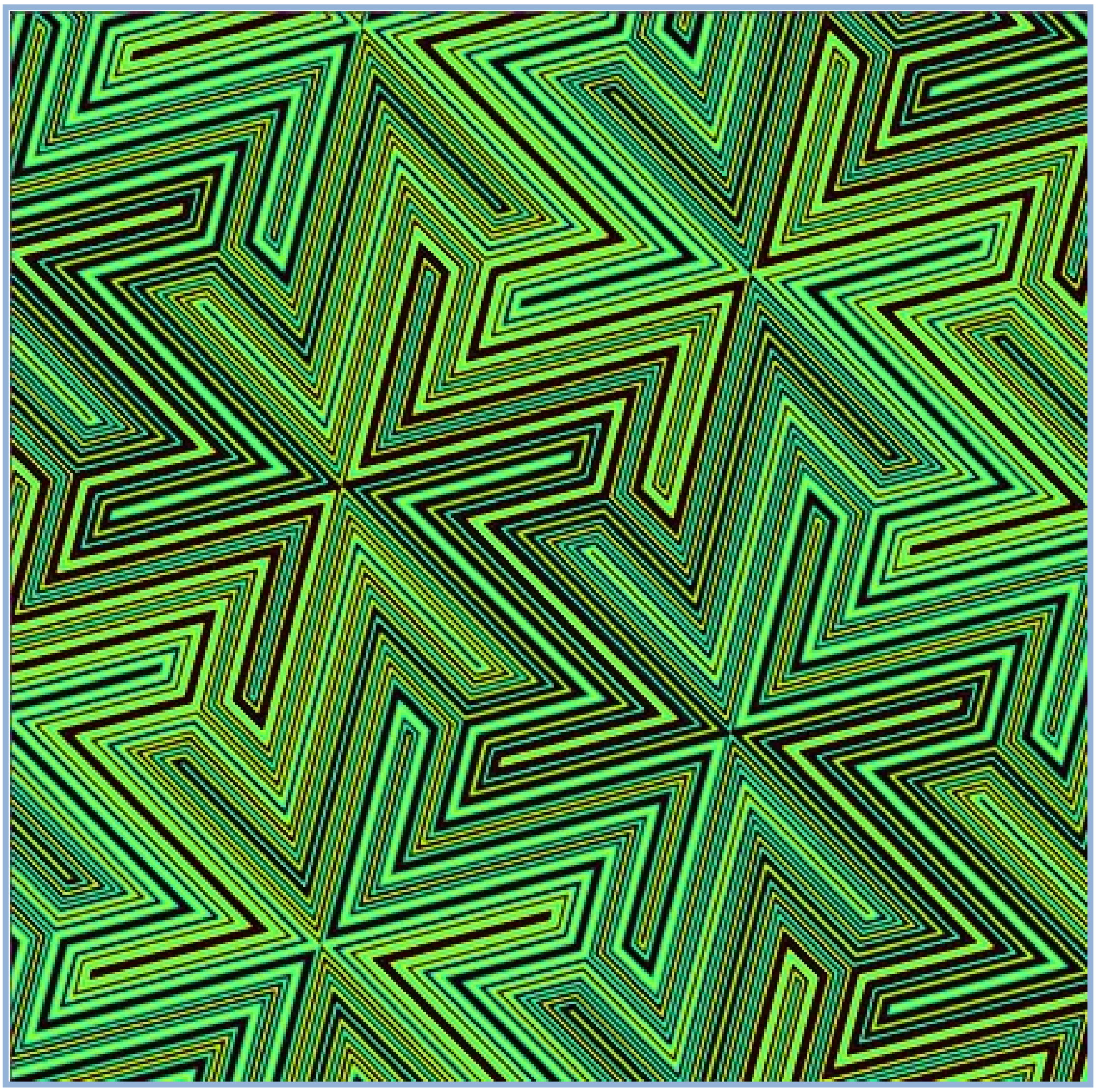


***Figure77:** Stripe path pattern with 7 levels in spiral winding, all at a marginal level.*

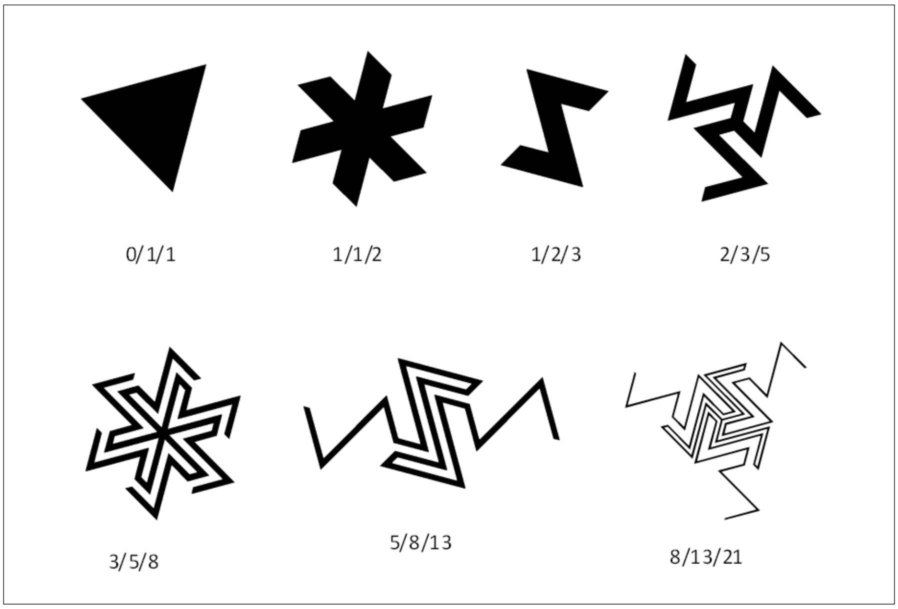


Figure 78: *The seven molds around which spiral winding takes place in stripe path [1,1,1,1,1,1,2].*

Layered color mixtures.

We can also generate beauty in these patterns by realizing layered color mixtures. Again based on the manipulation of the values n. Only coloring schedules with three or more colors are relevant here. The possibilities for such schedules depend on the type of symmetry and on the foldness of the rotation axis in the centre of the stripe path. Figure 79 shows the most current coloring schedules in p6: 2 schedules with three colors and 2 schedules with four colors.

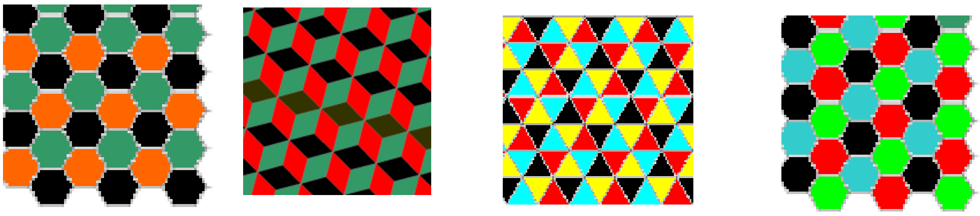


Figure 79*: Different coloring schedules in p6: 2 with three and 2 with four different colors.*

Starting from such coloring schedules two color mixing effects can be effectuated:

1. A *restricted number of colors* is involved in spiral winding around each center**.** The left pattern in Figure 80 is an example.
2. *One of the colors is dominant* in the spiral winding around each center. When the center of a stripe path is also the center of spiral winding, this stripe path is involved in the spiral winding directly around its center for roughly half of the surface while the other stripe paths involved 'account for' the other half. The right pattern in Figure 80is an example, where the other involved spirals deliver three of the four colors to (roughly) the second half of the surface.

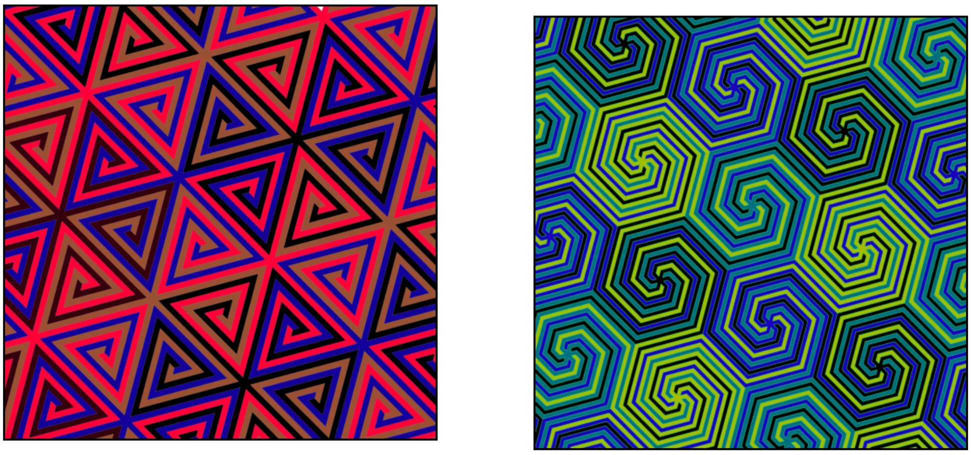
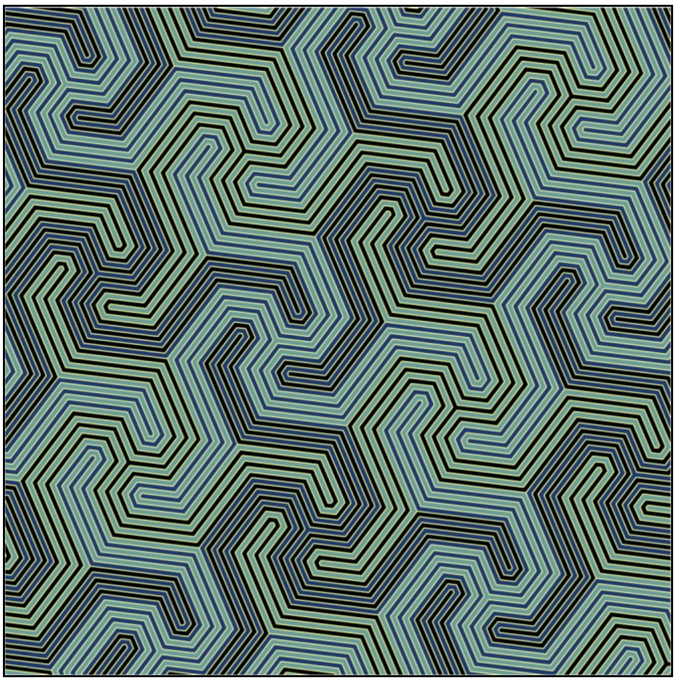


Figure 80: *Two methods to get color mixing effects*

By using one or both of the just described methods to achieve color mixing effects, beautiful layered color mixtures can be realized, which are expression of the beauty and elegance of Christoffel



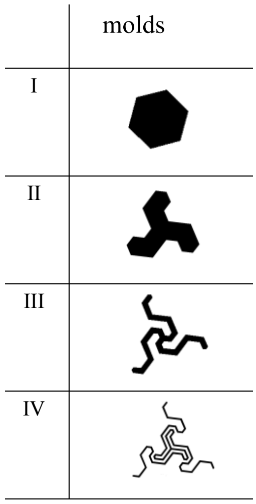


***Figure 81** : Layered color mixing in p3*

sequences.The pattern in Figure 81 is generated by the ratio is 12/29/41, consisting of four layers: [3, 2, 2, 2]. For   each of the four layers the mold for spiral winding is shown. In layer I and III the rotation axis of the mold coincides with the rotation axes of the final spiral. In principle color mixing takes place according to method 2 then. But in p3 the other stripe paths together deliver but one other color to the second half of the surface. Consequently there is no dominant role for the stripe path which delivers its color to the first half of the surface. Thus color mixing takes place according to method 1. At level III this means that there are three mixtures: green mixes with black, green mixes with blue and black mixes with blue. At level I this means that these mixtures themselves are involved in mixing according to a mixing scheme of two out of three: green/black with green/blue, green/blue with blue/black and green/black with blue/black.

More complex is the color mixing in Figure 82. Here color mixing occurs because there is spiral winding around 2-fold axes of color bundles which are delivered by spiral winding around threefold axes in which different combinations of 3 out of 4 colors are involved. The color mixing effect at level II is based on method 1; that of level I on method 1 and 2.

	molds
I	
II	

Figure 82: *More complex color mixing in p6.*

5 Crystals

5.1 intersection patterns in grids

Through the years I discovered that the beauty of Christoffel sequences, besides in layered spirals, also can be made explicit in crystal like figures. It's based on the introduction of a direction in a plane symmetry. We did that already in section 3. But now this direction is introduced in plane symmetries with higher order rotations: p4 and p6. At both ends of a *direction representing line segment* lies a rotation axis. The line segment shows up in more than one orientation: in two orientations in p4 and in three orientations in p6. By repeating the line segment in a systematic way, according to the constraints of the respective plane symmetry, a grid of triangles (p6) or squares (p4) arises. Figure 83 shows these grids for the direction ratio s/ℓ = 1/2. A 'hidden' structure of mutually intersecting lines is inherent to these grids. Let's have a closer look at these structures, focusing on the grid in p4. An arbitrary direction ratio *s/l* results in $s^2 + l^2$ intersections of a each line segments, by line segments in the other orientation. So in the example there are 5 intersections. These intersections can be conceived as the result of a series of shifts (Figure 84), which are alternately in horizontal and vertical direction. The rotation axis at the upper end of the vertical line at the start[42] of the shift process coincides with the rotation axis at the left end of the horizontal line segment. Moving from left to right, the horizontal shift is over a constant distance; the shifts in vertical direction alternately take place in U(pward) and D(ownward) direction and generate an D/U-series[43]. We can derive the *u/d*-ratio from the *s/l*-ratio in a very simple way[44]. The ratio between the upward shift and the total length of the line segment is *u/(u+d)*. This *u/(u+d)* ratio corresponds with a series of partial quotients which is twice the series of partial quotients of the s/ℓ-ratio , but in such a way that the first half is the reverse of the second half. When for example s/ℓ = [2,4] then *u/(u+d)* = [4,2,2,4]. That implies that *u/d* = [3,2,2,4], which is 22/75.

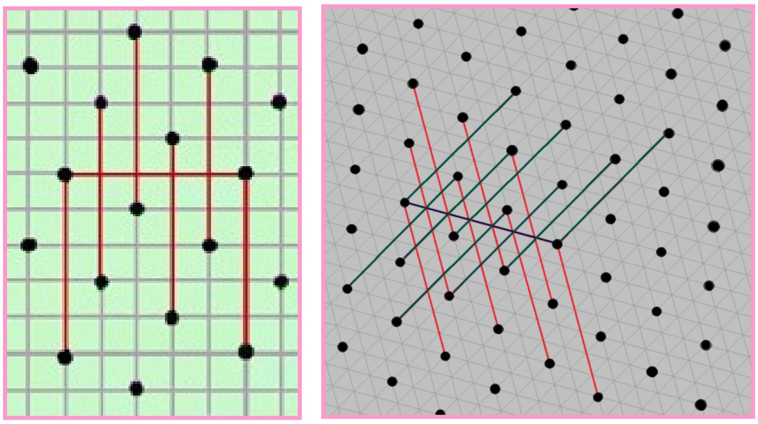


Figure 83: *Intersection pattern in p4 and p6*

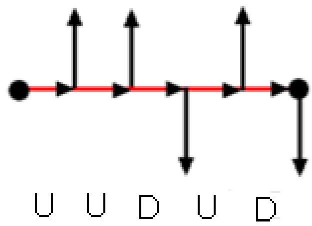


Figure 84: *Succession of upward and downward shifts.*

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[42] Focusing on one period of the intersection pattern that has its beginning in a rotation axis.

[43] In the plane symmetries p4 and p6, interaction patterns show up in different orientations. So the terms 'upward' and 'downward' must be interpreted in a relative way. In essence the term 'upward' refers to the shortest shift distance and the term downward to the longest shift distance. So the letters *s* and *l* should be more suitable initials, But we uses these last two letters already to specify the direction ratio *s/l*.

[44] The u/d ratio refers to two matters: 1) The number of upward intersections relative to the number of downward moving intersections 2) The shift distance of upward moving intersections relative to the distance of downward moving intersections. In the first case we write the variable-representing-letters in the ratio in capitals. The value of u and d interchange when we switch from the one to the other. When there are for example 12 downward moving intersections and 17 upward moving intersections, the shift distance of the upward moving intersections is 12 and that of the downward moving intersection 17. And vice versa. When the ratio between the shift distance of the two shift directions is meant we write u/d ; when the ratio between the occurrence of the two shift directions is meant we write D/U. (So in the term D/U-series d and u are written in capitals.) In both cases the respective ratio is a so called 'proper fraction' of which the numerator and denominator are co-prime.

intersection pattern is layered

The above intersection patterns are layered and have the character of a Christoffel sequence. Here too, the successive layers are represented by the partial quotients in the continued fraction of u/d. Figure 85 shows the example u/d = 12/17 of which the partial quotients are [1,2,2,2].

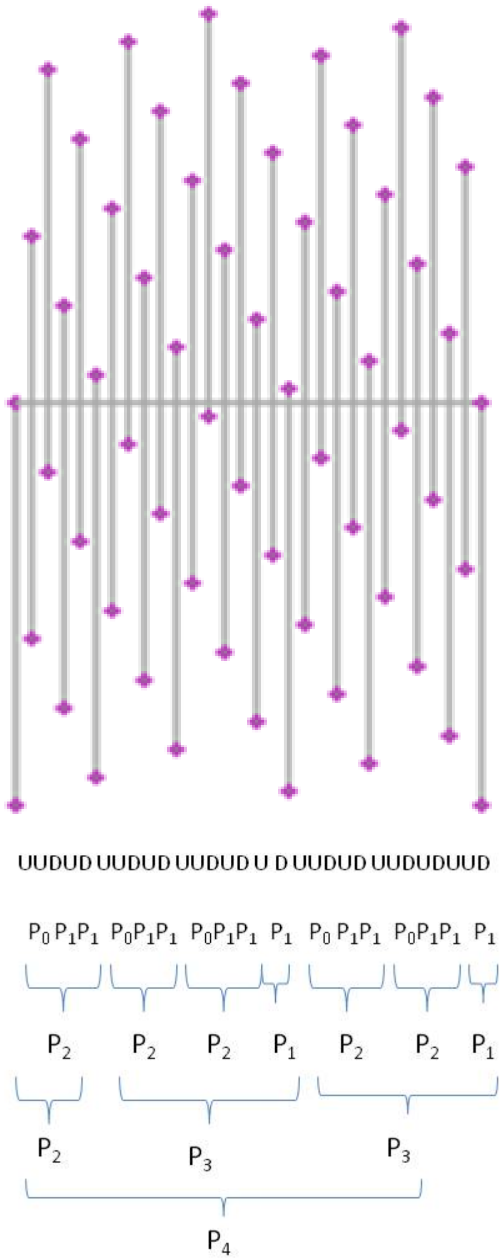


***Figure 85**: Intersection pattern of u/d = 12/17. To this pattern a Christoffel sequence with 4 layers is inherent .*

The layered structure becomes immediately explicit when we convert the dispersion pattern of the rotation axes in a structure of stacked triangles (Figure 86). By the interpretation of this structure in terms of the properties of Christoffel sequences, the reader should take the diagram in Figure 18 as frame of reference. The different colors represent the different layers. A triangle with a certain color represents a major at that layer. They are interrupted by minors that have the color of the preceding layer, where they are the majors. The base lines of the majors at a certain layer are the building blocks in the less steep sides of the triangles which are the majors at the next layer and the baselines of the minors at that level coincide with the more steep side of the majors at that next level. So the composition of a major at a certain layer out of one or more majors and one minor of the preceding layer is immediately clear. The number of line segments in the less slanting side of a triangle at a certain layer represents the value n at the next layer and the more slanting side of the triangles at the next level represents the value 1 in the alternation of n and 1 at that level. At layer 0, which doesn't represent a value n, the triangles are reduced to single lines that coincides with the sides of the yellow triangle. The majors (P_0) are represented by the successive line segments[45] in the less steep side of this triangle, that incidentally are followed by an additional line segment that coincides with the more steep side of the green triangles. The minors (P_{-}) here are represented by the more steep side of the yellow triangle. The majors are upward running while the minors are downward running.

[45] In this example there's but one segment coinciding with the steep side of the yellow triangle , because n_1 has value 1.The number increases when the value of n_1 increases. .

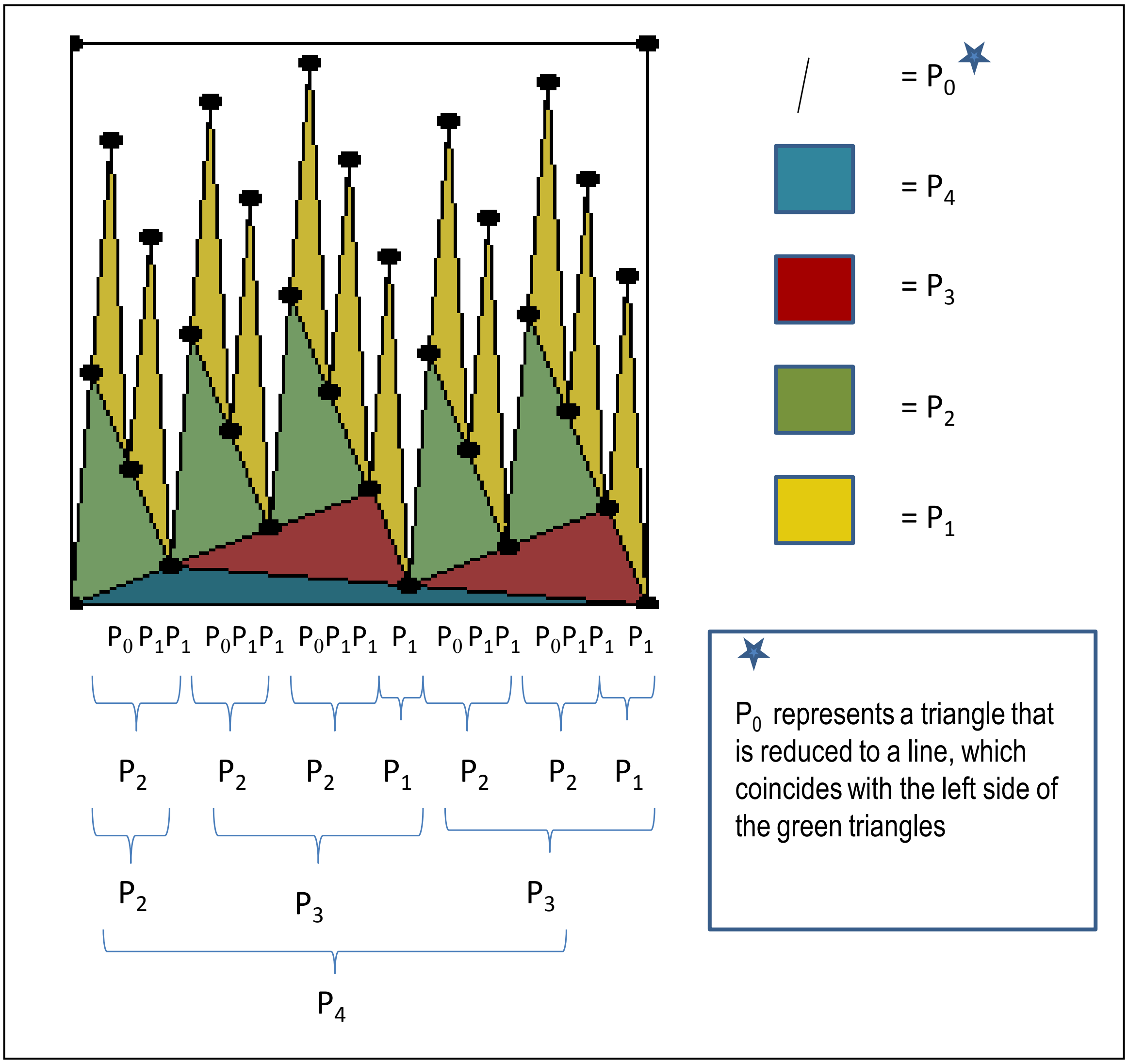
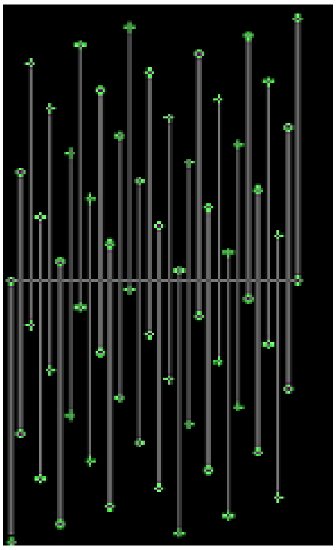


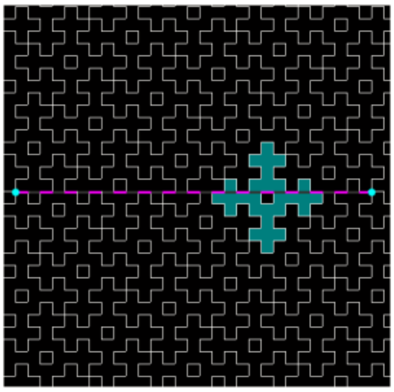
Figure 86 *: Layered nature of the shift pattern can be made explicit in a geometrical appearance by converting the dispersion pattern of the rotation axes in a structure of stacked triangles.*

5.2 Generation of crystals

The intersection pattern of directions has a certain complexity and one might think it must be possible to convert that complexity in geometrical shapes. At first I had not the faintest notion how to bring such form effects about. At long last, one day in 1989, I was lucky. I had bought a Commodore home-computer for a reasonable price and with the aid of it I developed a program for the introduction of the



a



b

Figure 87*: Converting intersection complexity to crystal figures.*

direction representing line segment in p4 and p6, with a possibility to eliminate parts of that line. In the beginning, nothing I tried did work. But then, in the small hours of a certain day, after a hectic night of trial and error, when I started to feel that I had lost all control and was randomly trying everything I could think of, I happened to introduce the direction representing line segment as a 'dotted line' [46] and all at once things began to happen: all sorts of fractal-like figures came tumbling over my computer screen. I could hardly believe my eyes. Every time I added a link the continued fraction, another layer was added to the figure complexity. I was deeply impressed by this result. Here, so I felt, the depth of God's Creation was unfolding on my computer screen. This was the kind of result I always had dreamed of. And when I succeeded in realizing similar results in p6 (Figure 88) , my euphoria once again reached unequalled levels.

In the following days and weeks I tried to share my feeling of ecstasy with many people: my family, the secretary at my work, my former boss, a professor in mathematics, my best friends. But no one was as deeply impressed by this crown on my life's work as I was myself.

At that time I was acquainted within an artist who had made the category of 'non-natural integers' the main subject of her artistic creations. I made an appointment with her and bombarded her with the news of my invention. But in that esoteric meeting we had too long a distance to bridge and we were unable to find each other in the matter that was so very important to both of us. So I went home, prone to a feeling of great disappointment.

[46] Current terms in language doesn't provide in possibilities for an accurate denomination. Meant is line that is divided in a number of equal pieces of which successively one is drawn , the next is not drawn and the one after that is again drawn.

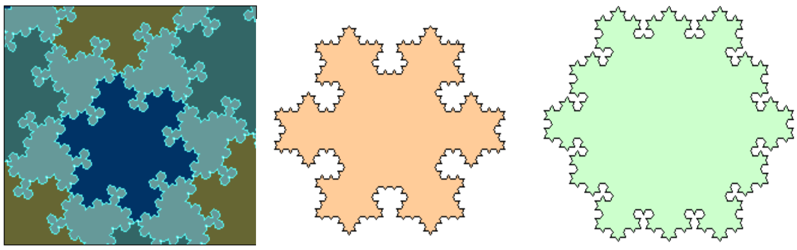


Figure 88 *: In the pattern at left, a "snowflake-like" figure occurs (in blue), which is composed of six (incomplete) "stars" , the "peaks" (peaks at the first stratum) of which are covered with smaller peaks (peaks at the second stratum) . The structure of the figure is determined by two a-values in the continued fraction, which both are 5. When the first value is increased (pattern in the mid), the number of "peaks" at the second stratum increases . When the second value is increased (pattern at right), the number of "stars" increases .*

relation between partial quotients and crystal shape

The diversity in the resulting crystals is directly related to the series of partial quotients in the *s/l*-ratio[47]. Every *s/l*-ratio results in an unique crystal shape, the characteristics of which can be derived from the respective series of partial quotients or '*a*-values'. On the basic level, every value in the series of *a*-values[48] is 2 and yields an extra level in the 'crossing-up' of the perimeter of the square (Figure 89a). If a value *a* becomes greater than 2, provided that it remains even, repetition occurs in the perimeter of the figure. We can illustrate this for the crystal figure represented by 2 2 2 . If we increase

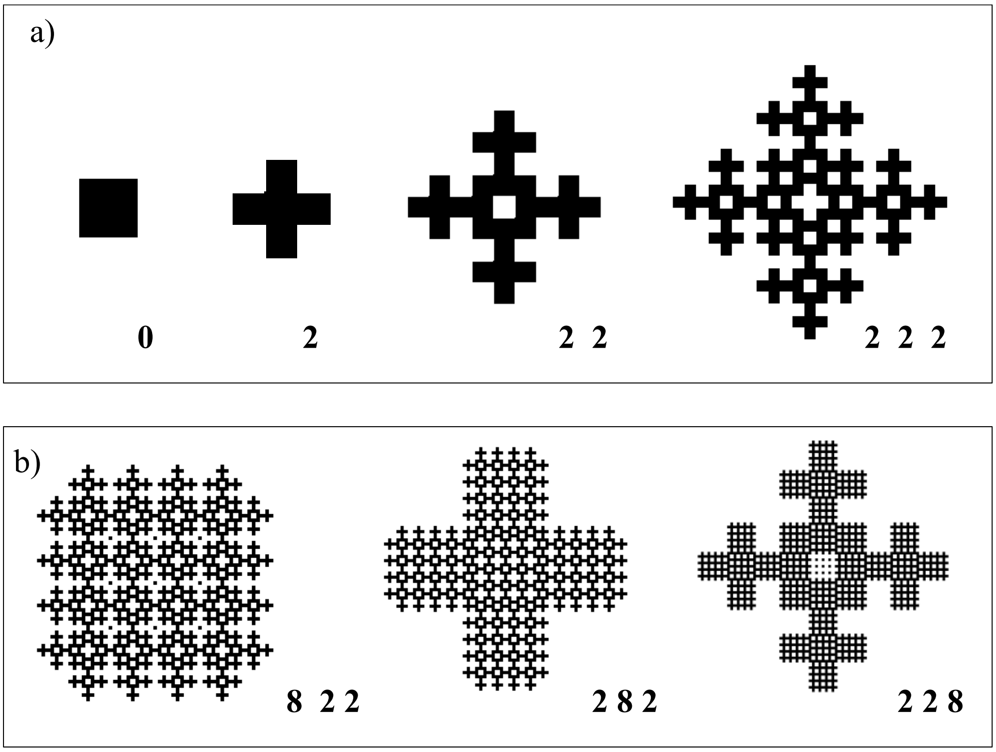


Figure 89*: Relation between a-values and crystal figures.*

[47] As we shall discuss further on, although the intersection pattern is based on the u/d-ratio, the partial quotients of the *s/l*-ratio fully covers the structure complexity.

[48] See Figure 15 for the notation system that is current for series of partial quotients.

the first value 2, all three layers in 'crossing up' are involved in repetition (Figure 89b). If we increase the second value 2, only two layers are involved in repetition. And if we increase the third, only one level is involved.
As soon as a value in the series of *a*-values is odd, the process of 'crossing up' stops and ring formation starts (Figure 90). That's because at such moment the value of *u* turns from even into odd. The size of that first odd *a*- value determines the number of columns in the ring formation. The next *a*-value after that odd determines the number of rings per center. After these two *a*-values which determine the characteristics of the ring formation, subsequent values in the *a*-series result in continuation of the process of 'crossing up' as long as these values result in an even value for u. But that crossing up then takes place at the level of the rings. As soon as subsequent values a result in an odd value for *u*, continuation of ring formation takes place[49]. Etc..

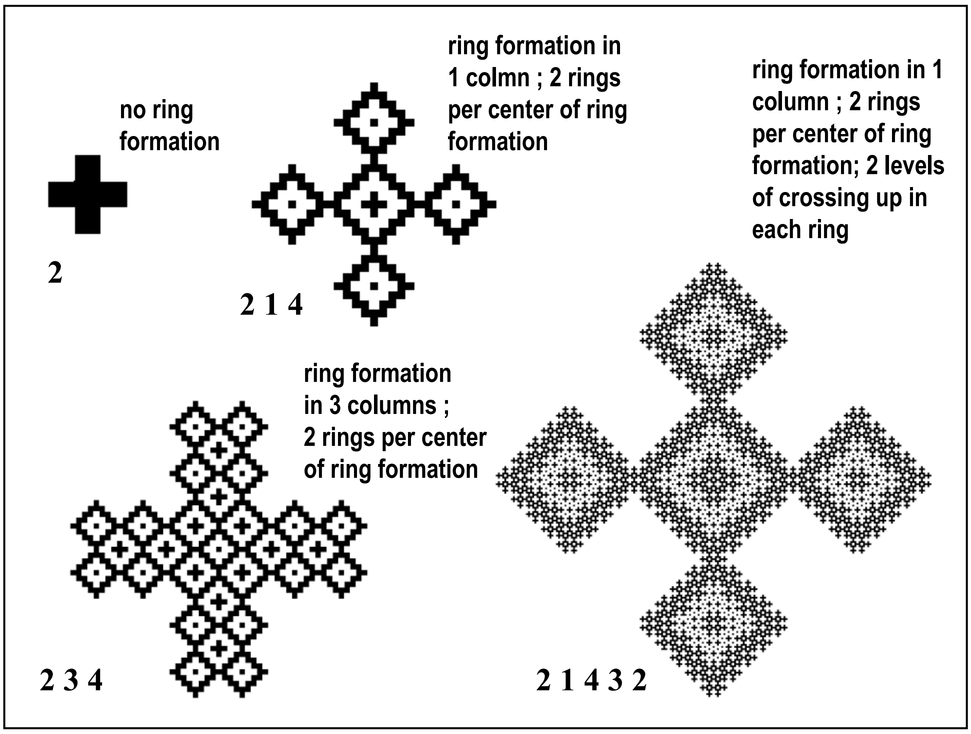


***Figure 90**: Ring formation due to odd values in the series of a-values.*

The layered character of Christoffel sequences not only becomes explicit in the outer border of the resulting crystals. It also becomes manifest in the embracement of lower order crystals by higher order crystals (Figure 91).

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[49] How the being odd or even of the successive a-values in the continued fraction of *s/l* is related to the being odd of even of the value u is rather complex and reaches outside the context of this paper. But it can be taken for granted that when only even a-values show up in the series, the resulting u-value is always even. But when for the first time an odd a-value shows, the resulting u-value turns into in odd . For example if *s/l* = [2] then u = 2, if *s/l* = [2.2] then u = 12 and if *s/l* = [2,2,2] then u =70. But if *s/l* is [2,2,3] then u = 99.

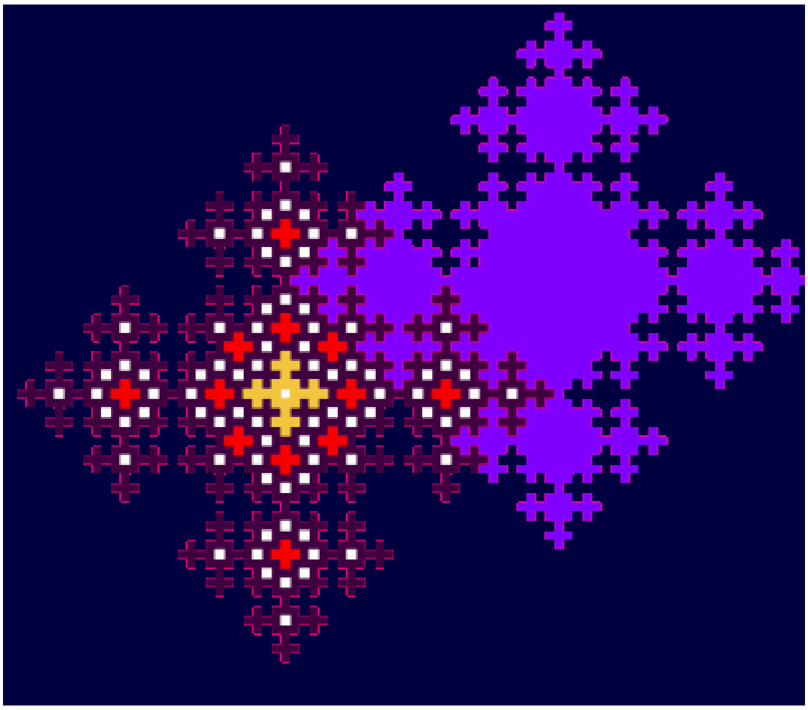


Figure 91*: Crystals within crystals.*

5.3 Understanding the built up of crystals

The euphoria about my crystals lasted only a few weeks. Then a long period followed in which nothing in life seemed more important than to understand the mathematical principles behind them. It should take me more than two decades. It was only recently, when I became acquainted with the observations and ideas of Elwin Christoffel, especially about the presence of a palindrome structure in 'his' sequences, that light definitely broke through. But let's forget that palindrome structure for a while and see what can be understood on a intuitive and global level .

5.3.1 The relative position of pieces in adjacent lines

At a global level, the generation of crystal patters can be understood in terms of the relative position of drawn (and not drawn) pieces in adjacent line segments. Pieces in two adjacent line segments can lie on the same level or on a different level (Figure 91a)[50]. That's applies to both orientations. In the interaction between the two orientations , three different building stones in the outline of crystals show up: squares, zigzag's and 'hat-like' elements (Figure 92b). Squares must be conceived as the most simple crystals. They are always embraces in the contours of more complex ones, as Figure 92c shows. This figure also shows that, when there 's but one level in the Christoffel sequence , the ordering of the two other elements in the outline of the 'embracing' crystals is simple [51]. One of two, being in minority, lies in the corners of that outline . The other, being in majority, repeats itself in the flanks of the outline. When there are more levels in the Christoffel sequence, the outline becomes more complex. Now still one of the element-types lies in the corners of the outline, but in the flanks both show up in a mixed and stratified way (Figure 92d).

[50] If u is even, then after an U-shift the pieces in the line segments left and right of the U-shift are lying on the same level (as far as the respective lines are running adjacent). After a D-shift those pieces lie on a different level. When u is odd, the reverse applies. In the example in Figure 92a, u is even.

[51] When values in the s/l-ratio increase, we must think in terms of more levels in embrcement.

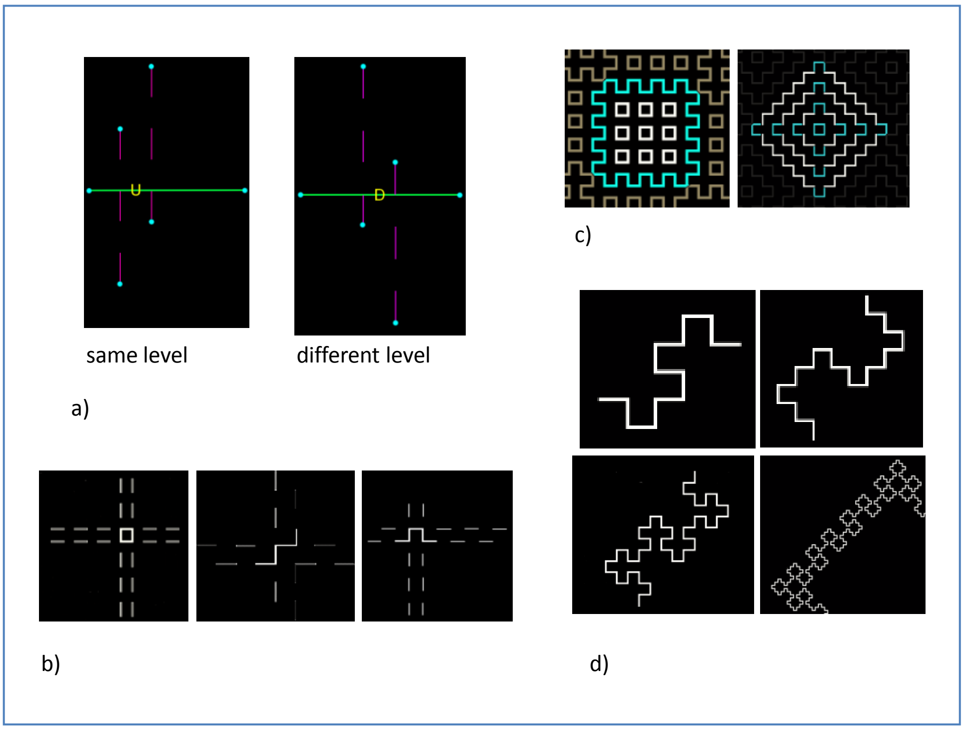


Figure 92 : Position of the pieces in adjacent lines deliver the building blocks for the outline crystals.

5.3.2 Understanding crystal built-up in terms of layered palindromes

The preceding is no more than a very global impression of how things 'work'. To understand the built up of crystals more fundamentally, we need to conceive them in terms of layered palindromes. Before we go deeper into that matter, we need to pay some attention to the notation system that we use to describe intersection patterns. May be the reader feels resistance to assimilate this system because of its apparently unnecessarily complicated 'virtual' character. But, you shall descry that it pays off.

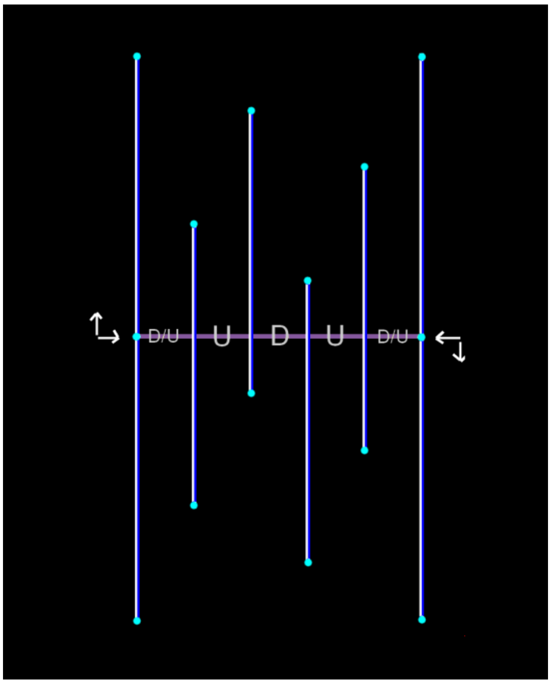


Figure 93: *Notation of U- and D-shifts.*

notation system

As already explained, we use the letters U and D to indicate U(pward) and D(ownward) shifts. These are the two elements that alternate in the Christoffel sequence. We position these letters in the mid of the pieces in which the intersected line segment is divided. Figure 93 shows the example *D/U*= 2/3. A letter U means that the next line segment, given the shift direction, moves upward relative to the preceding, in its intersection of the purple line. And a letter D means that that line segment moves downward relative to the preceding one. So the line segment is built up of 'U-pieces' and 'D-pieces'. Conform Christoffel's algorithm we get a letter U at the starting point of the successive intersections and a letter D at the end point. In plane symmetries however, it depends on the shift directions at which end of the intersected line we start the process of intersection, so the letters at the two extremities of the line segment interchange, dependent on the shift direction. To make the notation system independent of which shift directions are current, we can give the pieces at both ends as well an U- as a D-notation, as shown in the figure.

conceiving and describing the o-PAL within intersection patterns
Pitty for old gruff Elwin, it's the o-PAL that leads us to insight in the built-up of the crystals. To be possible to work with the concept of the o-PAL, and by consequence with the need to express letter halves in a geometrical way, we conceive each of the intersecting lines as composed of two halves. So in Figure 93 every intersecting line is presented as a double line of which one 'half' is white and the other ' half' blue. The letter U represents two successive line-halves (one blue and one white) between which an U-piece lie. And in the same way a letter D represents two successive line-halves between which a D-piece lies. So half a letter U is represented by one of the line halves and the half of the 'U-piece' that lies adjacent to it. For half a letter D applies the same, but now half a 'D-piece' lies adjacent to that line halve (Figure 94).

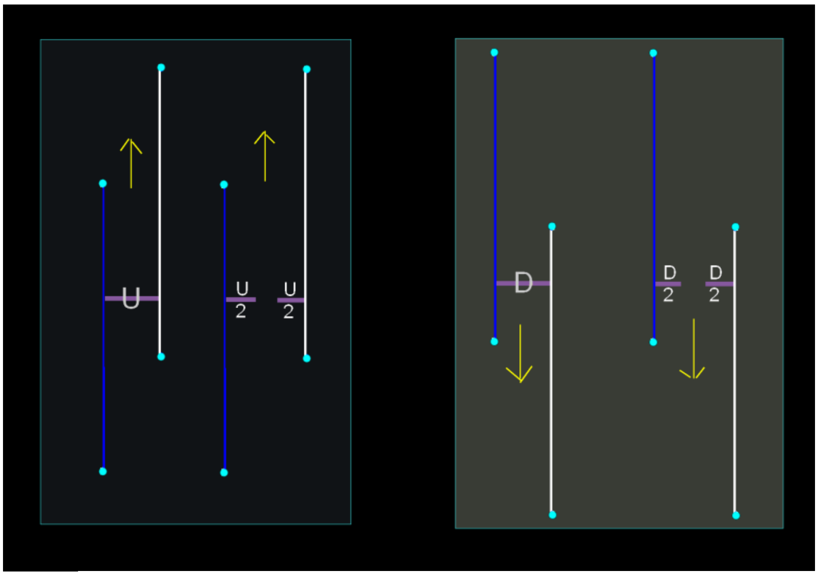


Figure 94: *Notation of o-PAL halves at basic level.*

positioning the r.p.'s
Now we have elaborated a notation system, we can position the r.p.'s more precisely in a geometrical way. As discussed in section 2, there occur standard 3 types of reversal points in a Christoffel sequence. One lies in the mid of an U-piece, one in the mid of a D-piece and one between two U-pieces that lie adjacent to each other. Figure 95 shows the locaton of these r.p.'s for the example $s/l =$ 1/2. The r.p. of the i-PAL lies in the mid of a D-piece. The r.p.'s of the o-PAL lie respectively between two U-pieces and in the mid of an U-piece [52].

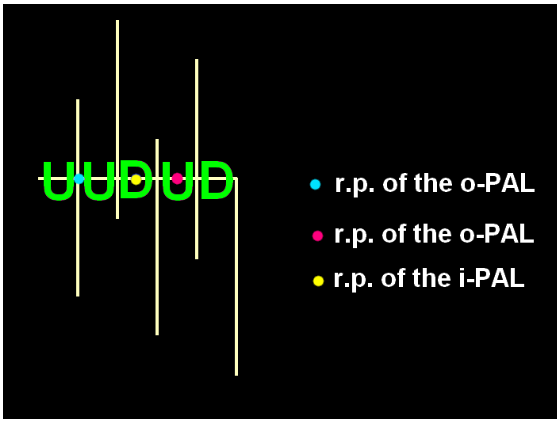


Figure 95 : *location of the r.p.'s in the o-PAL and i-PAL.*

[52] The location of the two r.p's in th o-PAL can only be made explicit when we have chosen which of the two U/D-pieces (at the two ends of the intersection sequence) we denote as an U-piece an which as a D-piece.

reversal within built-up of the intersection patterns

Before we can clarify the built-up of crystals in terms of layered palindromes, some attention must be paid to the reversal structure within intersection patterns. As stated before, although crystals arise from intersection patterns based on the shift-length ratio *u/d*, their shape can directly be derived of the series of the partial quotients in the direction ratio that we have named the *s/l*-ratio. That's not so surprising. The series of partial quotients of *u/d* roughly spoken can be conceived as a doubling of the series of partial quotients of *s/l.* So half of the partial quotients in the u/d series is predictable and does not deliver new structural information to the shaping of the crystals. Let's have a closer look at this.

The ratio between the upward shift and the total length of the line segment is *u/(u+d)*. This ratio corresponds with a series of partial quotients which is twice the series of partial quotients of the s/ℓ-ratio, but in such a way that the first half is the reverse of the second half. When for example s/ℓ = [2,4] then *u/(u+d)* = [4,2,2,4]. That implies that *u/d* = [3,2,2,4], which is 22/75.

We can find back the just described reversal in the series of partial quotients in the intersection pattern. To show this, let's go to a more simple example, namely *u/d* = [1,2 | 2,2]. The red line between the brackets indicates the reversion point in the partial quotients. When we see the two halves as each representing its own Christoffel sequence, we must reverse the order of the quotients in the second half[53]. The two halves roughly represent the same Christoffel sequence: [1,2] represents UUDUD and [2,2] represents UUUDUUD. Both sequences have two layers, but the partial quotient that represents the first layer is 1 less in the first sequence than in the second. So in the first sequence there occurs one U less between every two D's than in the second. The two sequences can be made explicit in a geometrical mode in the pattern of stacked triangles (Figure 96). The green triangles represent the majors at the reversion level and the yellow triangles represent the minors at this level. To come from the first rotation axis to the next , in the base line of the green triangles, an U(pward) shift is needed. But where a minor interrupts the concatenation process of the majors, a D(ownward) shift is needed to reach the next rotation axis in the base line of the green triangles. So the way in which majors and minors alternate at the reversion level can be described as UUUDUUD. It are the last two layers, the red and the blue, which determine the layered structure of this sequence. But this sequence, with one U less between every two D's, also describes the built-up of a single major at the reversion layer. In

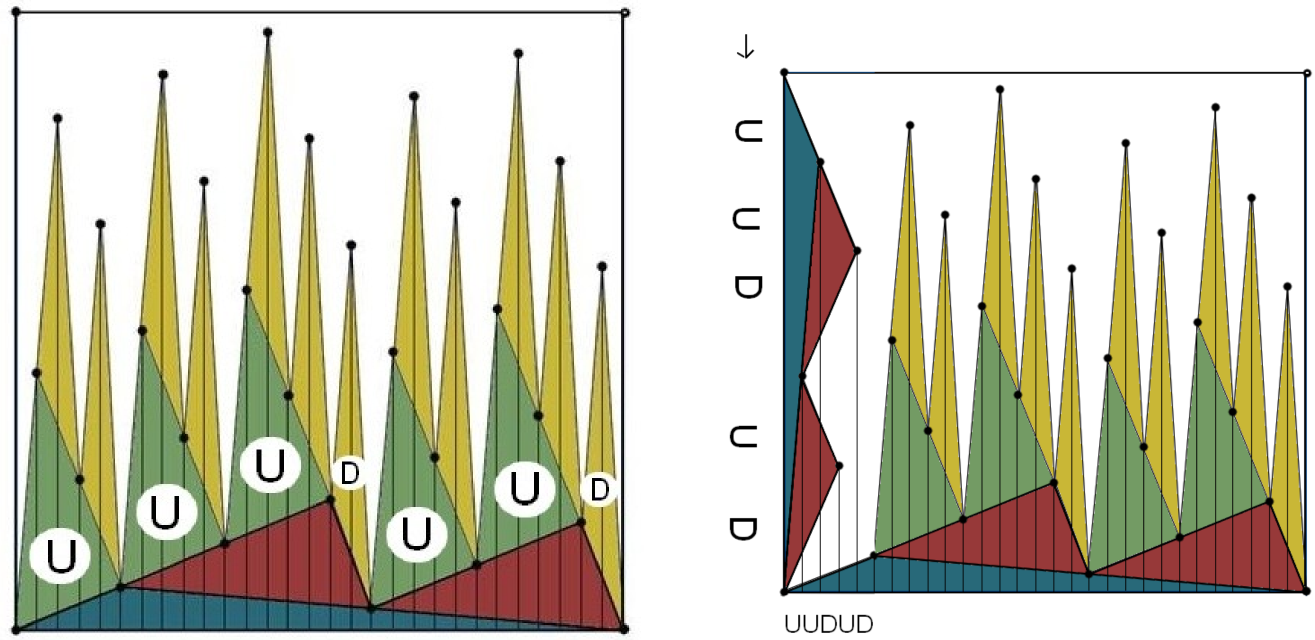


Figure 96: *Reversion in partial quotients declared in terms of the pattern of stacked triangles.*

the right part of the picture, the most left major is singled out and declared in terms of the red and blue triangles , that now are positioned in the other orientation. To declare the built up of that major, in the red triangles the value n must be reduced by 1. So the less slanting side of these triangles doesn't have two parts, but only one.

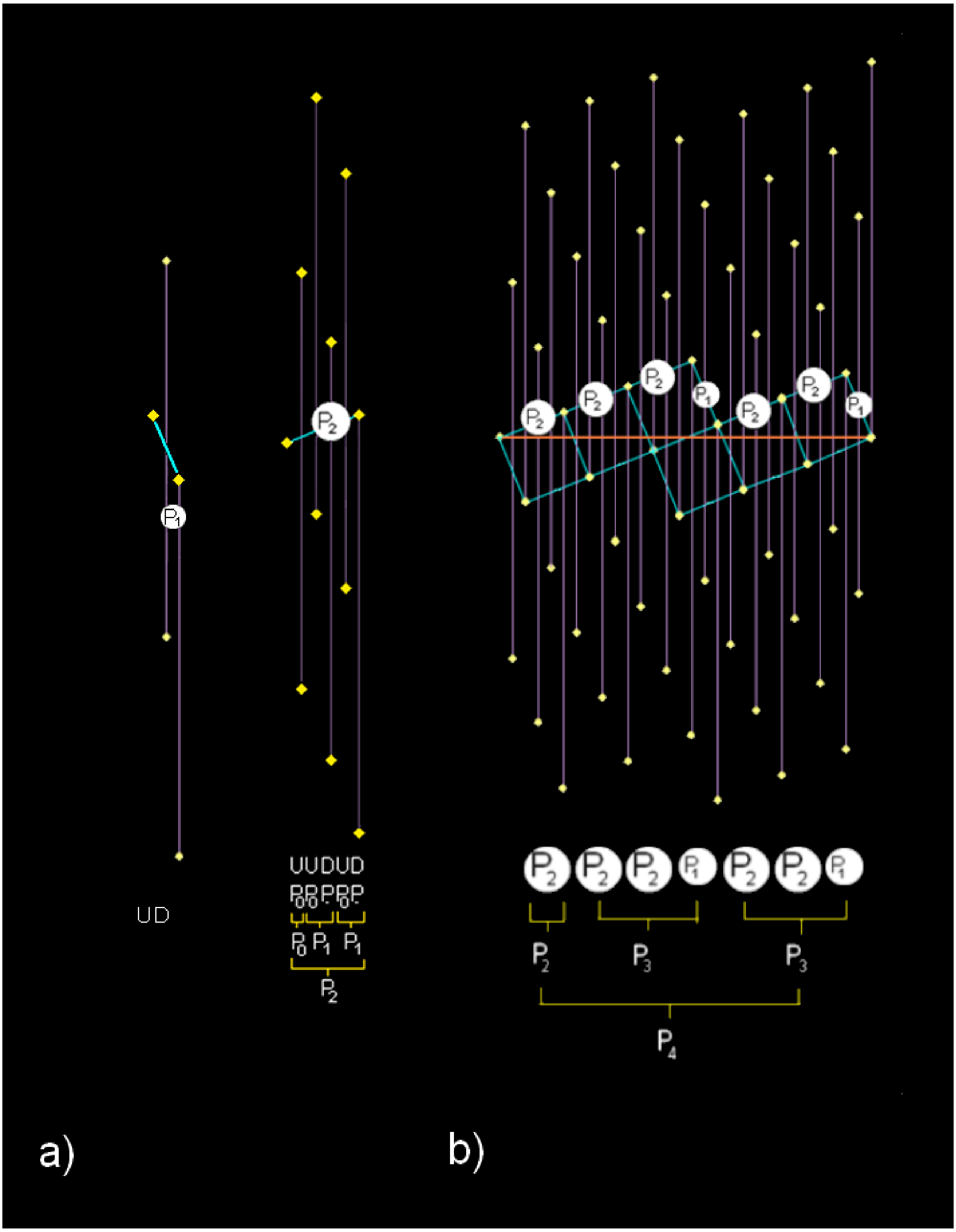


Figure 97: *How the reversion in the series of partial quotients manifests itself in the intersection pattern .*

Figure 97 tells it in other words. The left picture shows the built up of the major (P_2) and minor (P_1) at the reversal level in terms of U(ward) and D(ownward) shifts. For the major at this level a decomposition diagram is added, in which the built up until the reversal level is made explicit The right picture shows how the overall sequence is built up from the major and minor at the reversal level. Of this built up also the decomposition diagram is shown. The two diagrams are nearly the same. They only differ in the value n of the first layer. The number of P_0 's between every two P-'s is 2 while the number of P_2's between every two P_1s is 3. It means that the difference in the value n of the first layer inthe decomposition diagrams is 1.

palindromes

The built up of crystals matches with the just discussed phenomenon that there's always a reversion in the built up Christoffel sequences. As made clear, the crucial building blocks in that built-up are the minor and minor at the reversal level. In a crystal these building blocks heap up in a standard way, whereby these building blocks are delivered by Christoffel sequences in two orientations.

delivering of building blocks by one period of a Christoffel sequence in one orientation.
Before we go to that plane filling picture, let's first look at the 'delivering' of building blocks to crystals by one period of a Christoffel sequence in one orientation. Let's again take the example *s/l* = 2/5. Figure 98 shows such a period. It traverses a number of crystals, thereby delivering building blocks to them. The different building blocks are demarcated by the mirror axes through the successive crystals. They are the majors and minors at the reversion layer. The period delivers 5 majors and 2 minors to the crystals atthe reversal level.

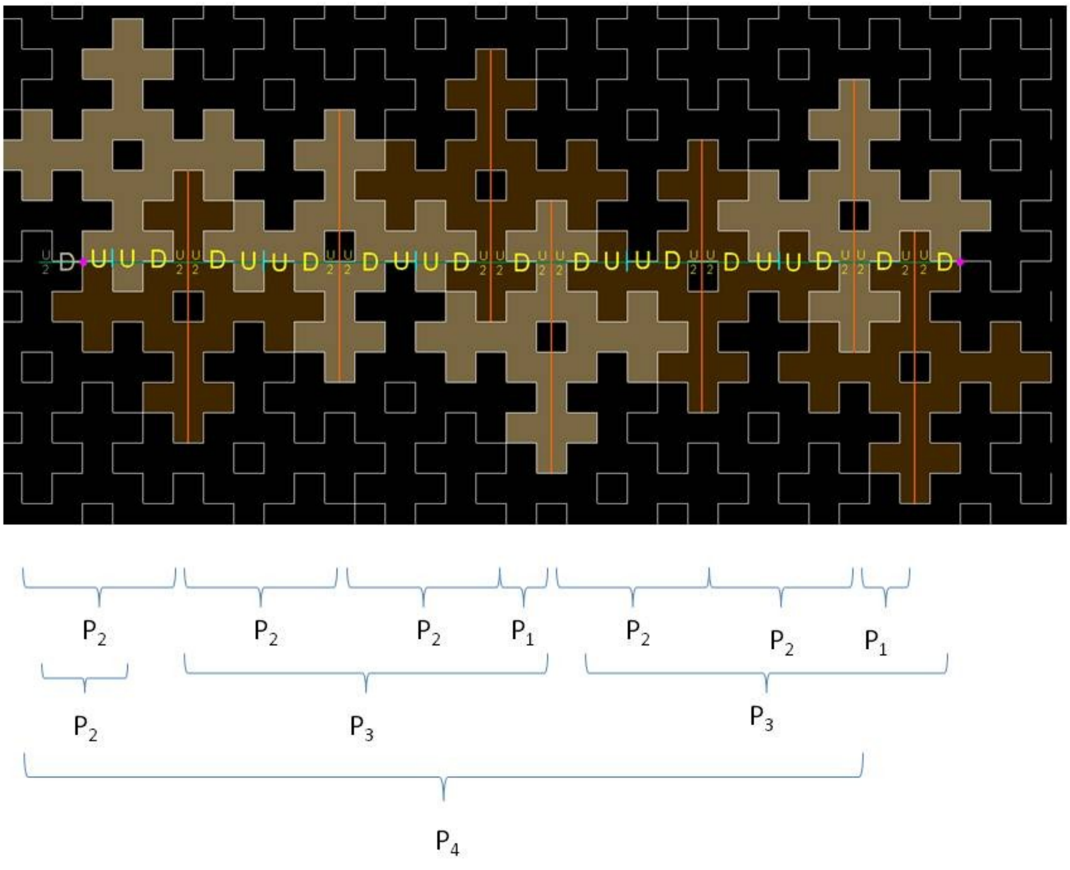
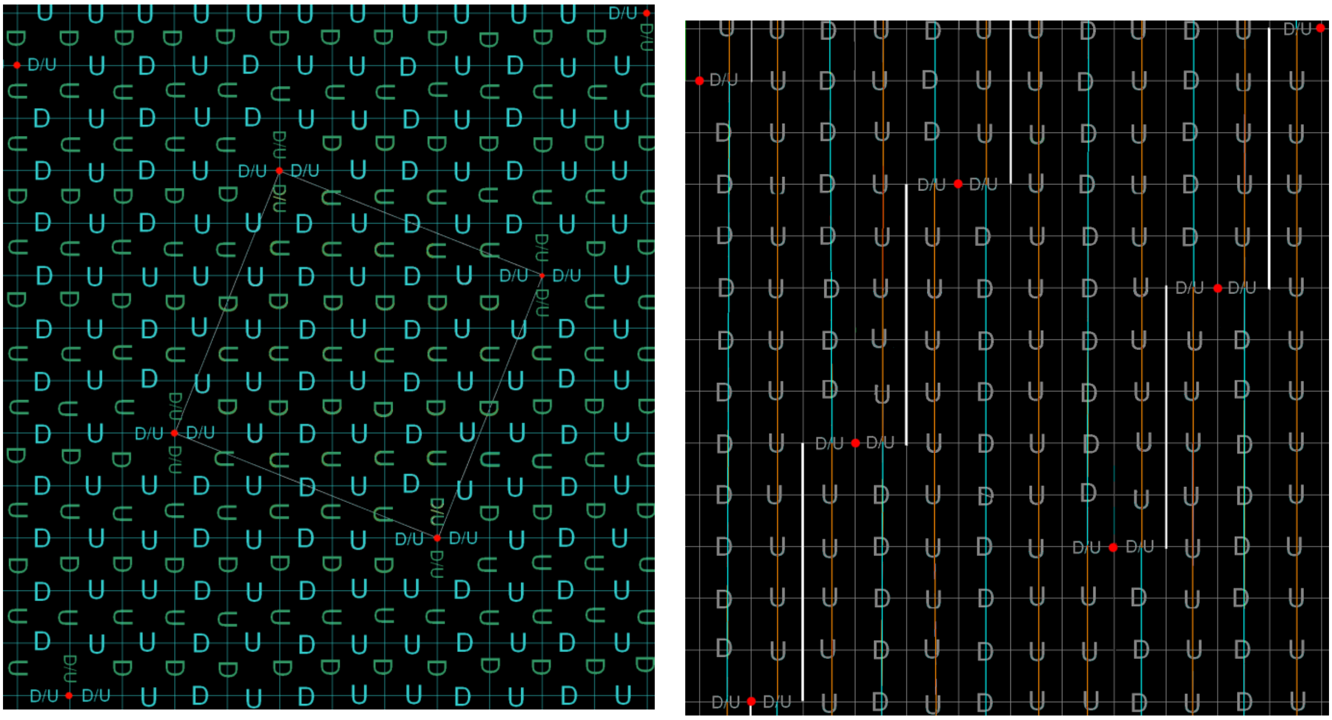


***Figure 98**: Delivering of building blocks by one period of the Christoffel sequence s/l = 2/5 , to the crystals that it traverses its path.*

In a the plane filling interaction of Christoffel sequence, different sequences interact in a plane filling way. Figure 99 a illustrates the example *s/l* = 2/5. The white outline between four red rotation axes demarcates one OTRU within this plane symmetry. It's clear that this basic shape of the OTRU doesn't give us a clear image of how the letter strings that represent o-PAL components heap up in spatial patterns.



a b

Figure 99 ***:*** *Plane filling interaction of Christoffel sequences*

To start with, we discuss this heaping-up in one orientation of the letter sequences . Figure 99b shows the arrangement of the letters in one orientation. On one dimension of the matrix (row wise[54]) they are arranged as Christoffel sequences. On the other dimension (column wise) all 17 letters U of one period are placed behind each other, followed by all 12 letters D. A multiplicity of r.p's is present in the letter pattern,. These are three different types: one type shows up in the mid of every U-piece, another in the mid of every D-piece and the third between every two U-pieces that lie directly adjacent to each other. The lines which are drawn in the letter pattern (column wise) connect r.p.'s of the same type: orange lines connect the r.p.s that lie in the mid of the U pieces, blue lines those that lie in the mid of the D-pieces and white lines those that lie between two U-pieces. We shall call these lines 'rp-lines'.

In discussing the heaping-up patterns of the o-PAL components we focus on the layers until the reversal level. In the example there are two of them[55]. In each, two types of columns show op: wider columns, in which the majors at that layer heap up, and narrower , in which the minors of that layer heap up (Figure 100) . The wider columns at layer 1 contains P_1 components and the narrower columns contain P_0 components. At layer 2 the wider columns contain P_2 components and the narrower columns P_1 components.

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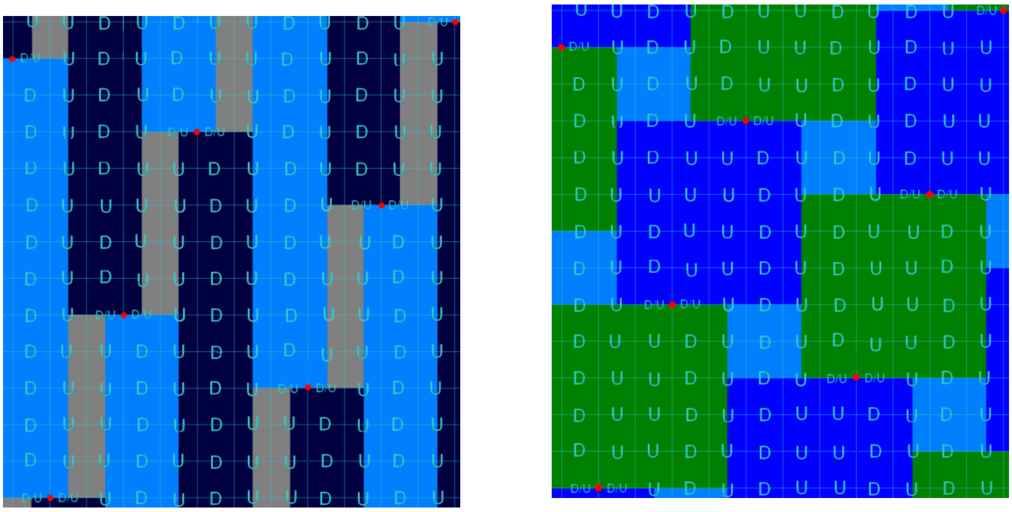
[54] The terms ' rowwise' and 'columnwise' are relative here.
[55] The basic layer 0 , at which the letters U and D, each standing on its own , are the (trivial) palindromes, where by the U's are the majors and the D's the minors, is disregarded here.



layer 1 layer 2

Figure 100: *Heaping-up-pattern of palindrome components (majors and minors) at layer 1 and layer 2. The example is s/l= 2/5.*

rp-lines in the heaping-up pattern

Let's look which role of the multiplicity of r.p.'s plays in the heaping-up patterns. We again take *s/l* = 2/5 , of which *u/d* = 12/17, as the example. So there are 34 r.p's active in every period of the Christoffel sequences. The general model in Figure 22 also applies to these r.p's, even though they are

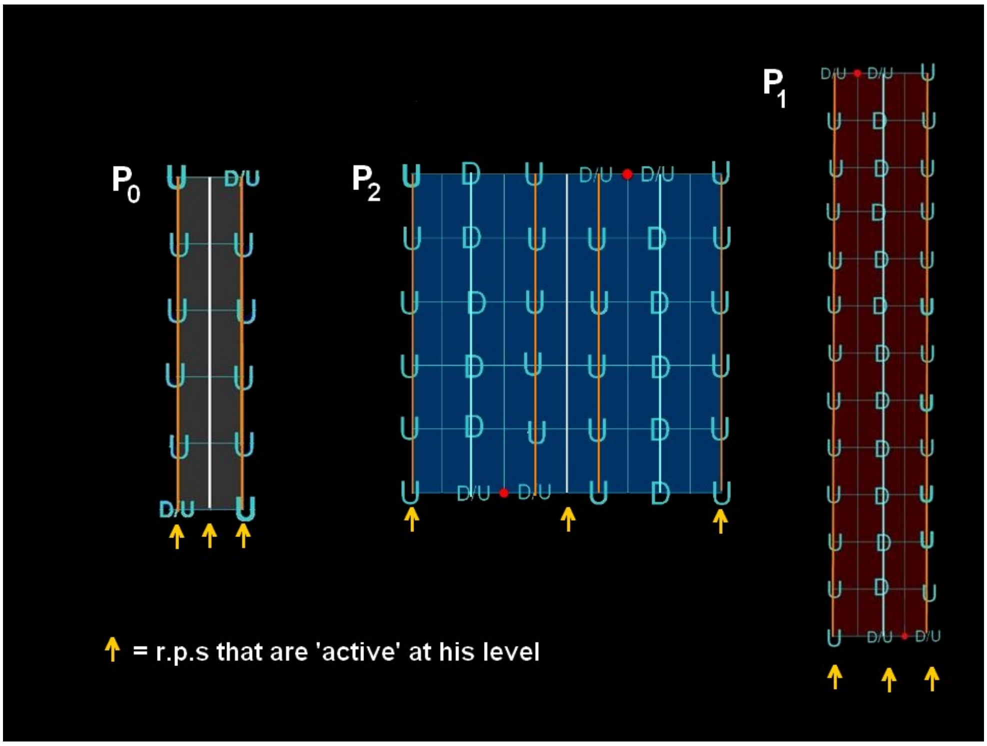


Figure 101*: Role of the three types of r.p.'s in the palindrome component columns of three different levels.*

embedded in spatial arrangements .The higher the layer, the less of the total number of r.p'd in a period is 'active'[56] there. On the other hand certain r.p.'s are involved at more than one level. Figure 101 hows the role of the 34 r.p.'s in the different columns at the two levels. The 5 r.p.'s that are located between two U-pieces and lie on the white lines are active in the column of P_0-components and in the column of P_2-components. The 12 r.p.'s that are located in the mid of D-pieces and lie on blue lines are 'active' in the column of P_1-components. The 17 r.p.'s that are located in the mid of U-pieces and lie on orange lines are active in the column with P_0-components and the column with P_2 components. In the column with P_2-components a number of r.p.'s located in the mid of U-pieces and all r.p.s located in the mid of D-pieces are not 'active'.

understanding crystal built-up in terms of rp-lines
Especially the orange rp-lines are crucial in the understanding the built-up of crystals. The r.p.'s on these lines always generate a mirror symmetry in the arrangement of the U and D pieces, which reaches out until the end of the palindrome components of equal level that lie directly adjacent to them. This mirror symmetry through the successive r.p.'s on these lines gives the crystals their characteristic shape. Figure 102 shows the example *s/l*=2/5. Adjacent to the rp-line through the center of the crystal lie o-PAL components of level 2,1 and 0. On either side of this center-line lies another rp-line. Adjacent to these lines lie o-PAL components of level 0. In this example there are three different levels , but there can be many more, dependent on the partial quotients in the *s/l* -ratio. The three orange line segments that we see within the crystal can be traced on the path that one orange line makes through the plane, thereby traversing different crystals. When we follow the whole path of one orange line, starting from the rotation axis that lies in the center of the crystal and walking over that line, step by step a reduction in the level of the r.p's takes place[57]. Let's have a closer look at this stepwise reduction in the level of the r.p's. Let's thereby focus a on two smaller and two larger squares lying adjacent to a line through the center ('center-line') of an OTRU. Figure 103 shows again the example *s/l = 2/5* . The arrangement of the 4 squares can be considered as the general model in the heaping up of majors and minors at the reversal level. The ratio between the length of the sides (= number of letters in rows and columns) of the squares in every case equals the direction ratio *s/l*. The majors and minors that heap up in the larger and smaller squares are coming from Christoffel sequences that lie just above each other in the
plane. Reasoning from the center of the OTRU and following the center-line, every next sequence that we traverse can be conceived as resulting from the preceding by carrying out a shift over distance u and one over distance 1 (see picture) . Every new sequence intersects the center line in a point that is an r.p. in the built up of its o-PAL's. The r.p.'s partly lie between palindrome components of the same order and partly between palindrome components of different order. When the level of the palindrome components differs, the mirror symmetry extends up to the end of the o-Pal component of the lowest level, which is also present directly adjacent to the rp-line within the o-PAL component of the highest order[58]. In Figure 103 there are 3 r.p.p.'s lying between P_2's and 4 r.p's lying between a P_2 and a P_1.

[56] With the term ' active' is meant that the respective r.p.'s at this level are reveral points between letter strings that represent o-PAL's The o-PAL's at both sides of the r.p. can have the same level of a different level, In the last case only for the extent of the lowest level oPAL, the respective r.p. is reversal point
[57] With the level of the rp is meant the level of the components for which they function as r.p.
[58] The component of the highest level in that case has components of the lowest level as its bolding blocks, which makes that there always can shows up a mirror symmetry between a pair of lowest order components around an r.p.. See also footnote 15 .

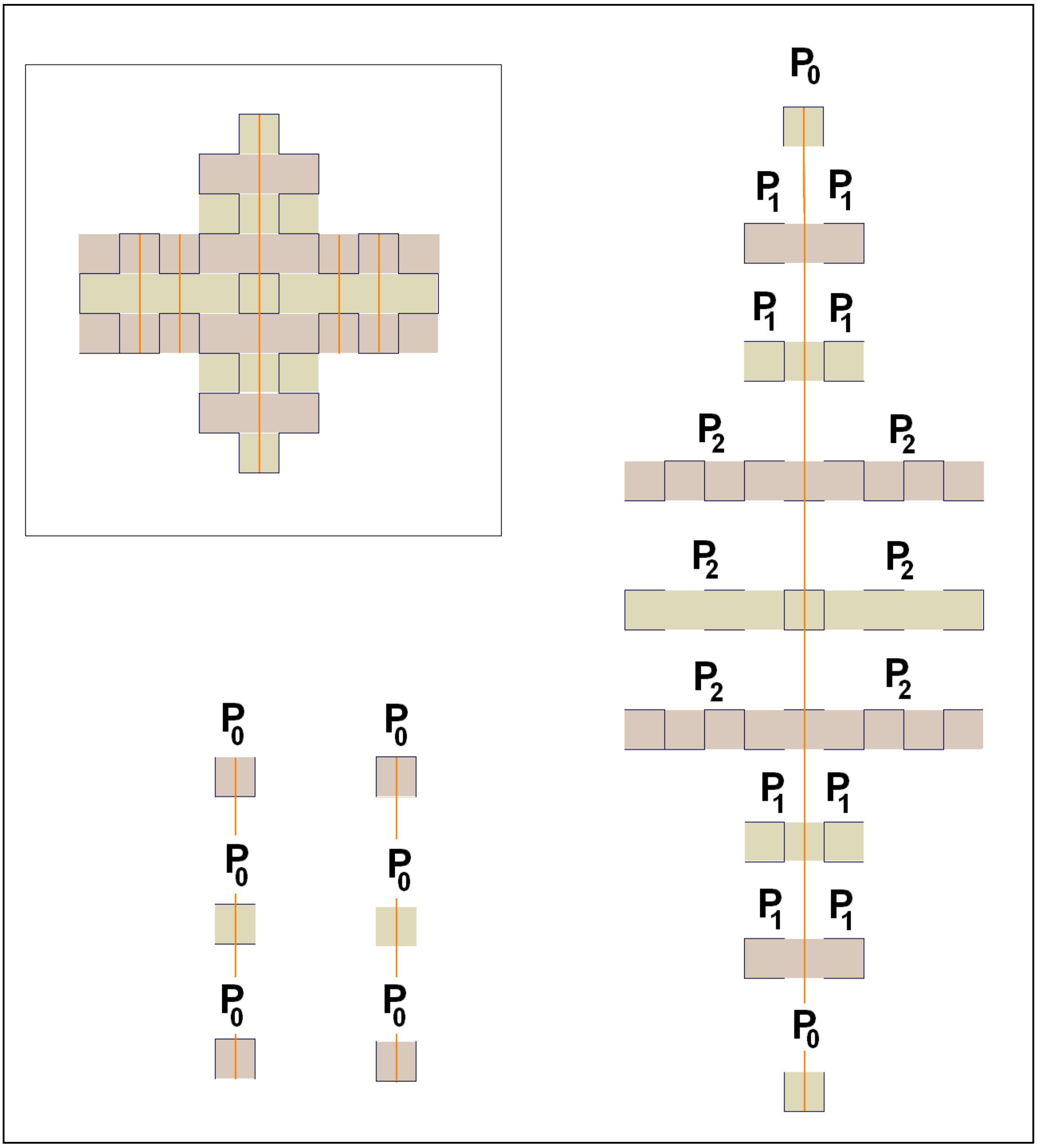


Figure 102: *Around the r.p.'s lying on the rp-lines that run through the mid of the U-pieces lie palindrome components of the same order , that together form a mirror symmetry. This symmetry reaches out until the end of these palindrome components.*

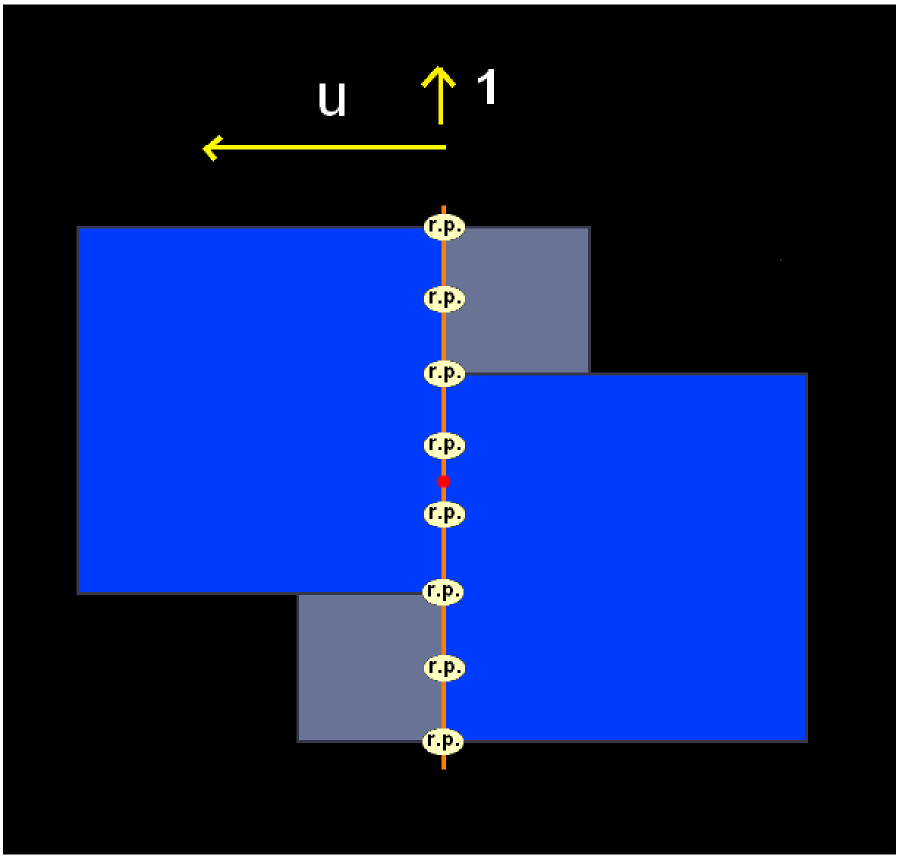


Figure 103: *General model of the heaping up of o-PAL components.*

This is consistent with the general model of the built-up of o-Pals as presented in Figure 22 in section 2. Figure 104 shows the elaboration of that model for s/l = 2/5. This figure makes clear that there are

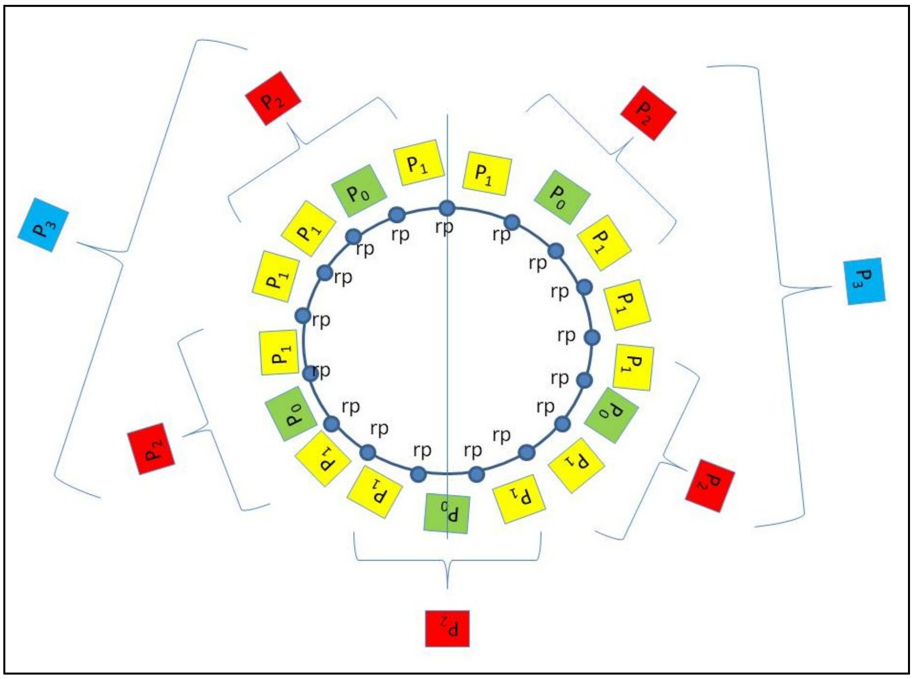
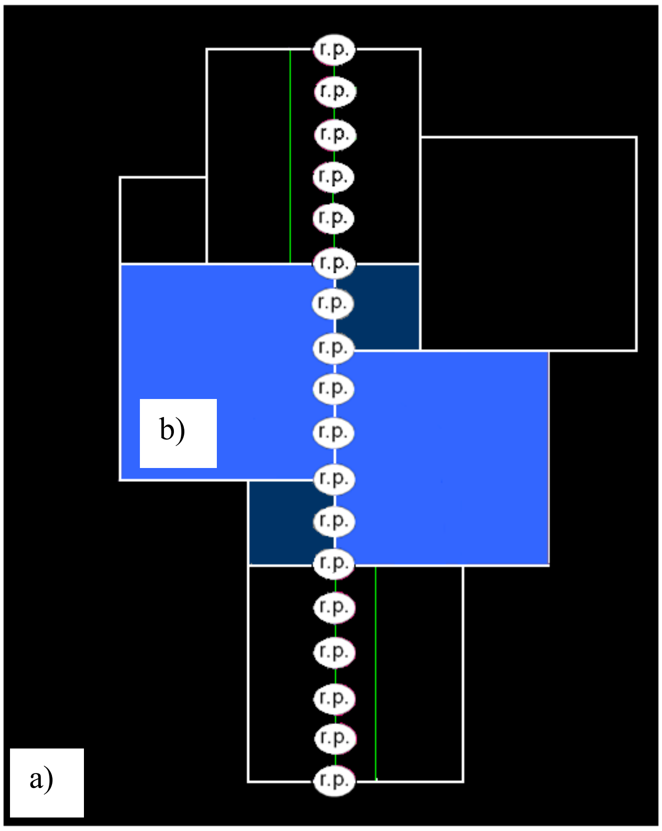


Figure 104: *Elaboration of the general model of the built -up of o-PAL's for s/l=2/5.*



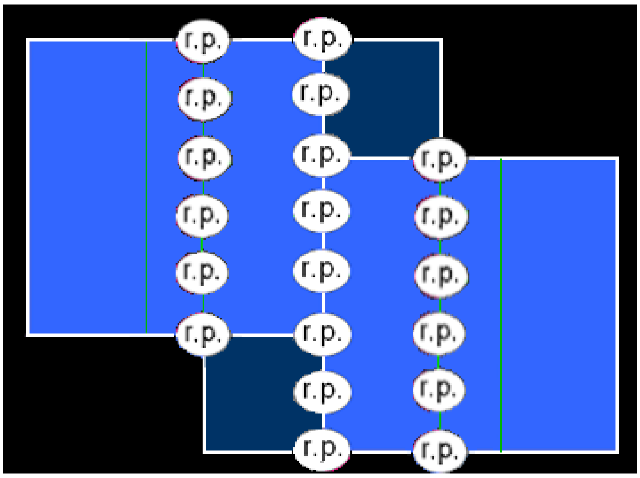


Figure 105: *We can follow the rp-line beyond the blue marked squares, where this line no longer borders larger and smaller squares but traverses them (a). The path that the rp-line makes then , can be displayed in the initial blue squares (b).*

also 10 r.p.'s that lie between and a P_0 and a P_1 component. We can find these 10 back on the center line, when we follow this line further, beyond the blue marked squares, where it no longer borders larger and smaller squares but traverses them (Figure 105a). The continuation of the path also can be displayed within the initial blue squares (Figure 105b). In the example, in which $l/s = 2/5$, the lines-segments that represent the continuation of the path, only go through the larger squares. When the values in the s/l ratio increase, also in the inside of the smaller square show up lines that represent the continuation of the path of the rp-line (see Figure 106).
The level of the r.p.'s on the lines that traverse smaller squares is always the same as that of the lines in the larger squares on which they connect. In the larger squares, the main part of the rp-lines do not connect to lines in smaller squares, but to lines in other larger squares. And the transition of one larger square to another, a reduction in the level of the r.p' s lying on the respective lines make takes place. Starting the path from the center, every time the line passes a border between two large squares, a reduction takes place in the range of the r.p. that successively lie on this line. When the values in the *s/l* ratio become larger, the diversity in the level of the r.p.'s becomes greater, and so of the number of locations at the borderline between large squares where the level of the r.p's diminishes in the transition of one square to the other. Figure 106 shows the level decreases for the $s/l = 5/12$.

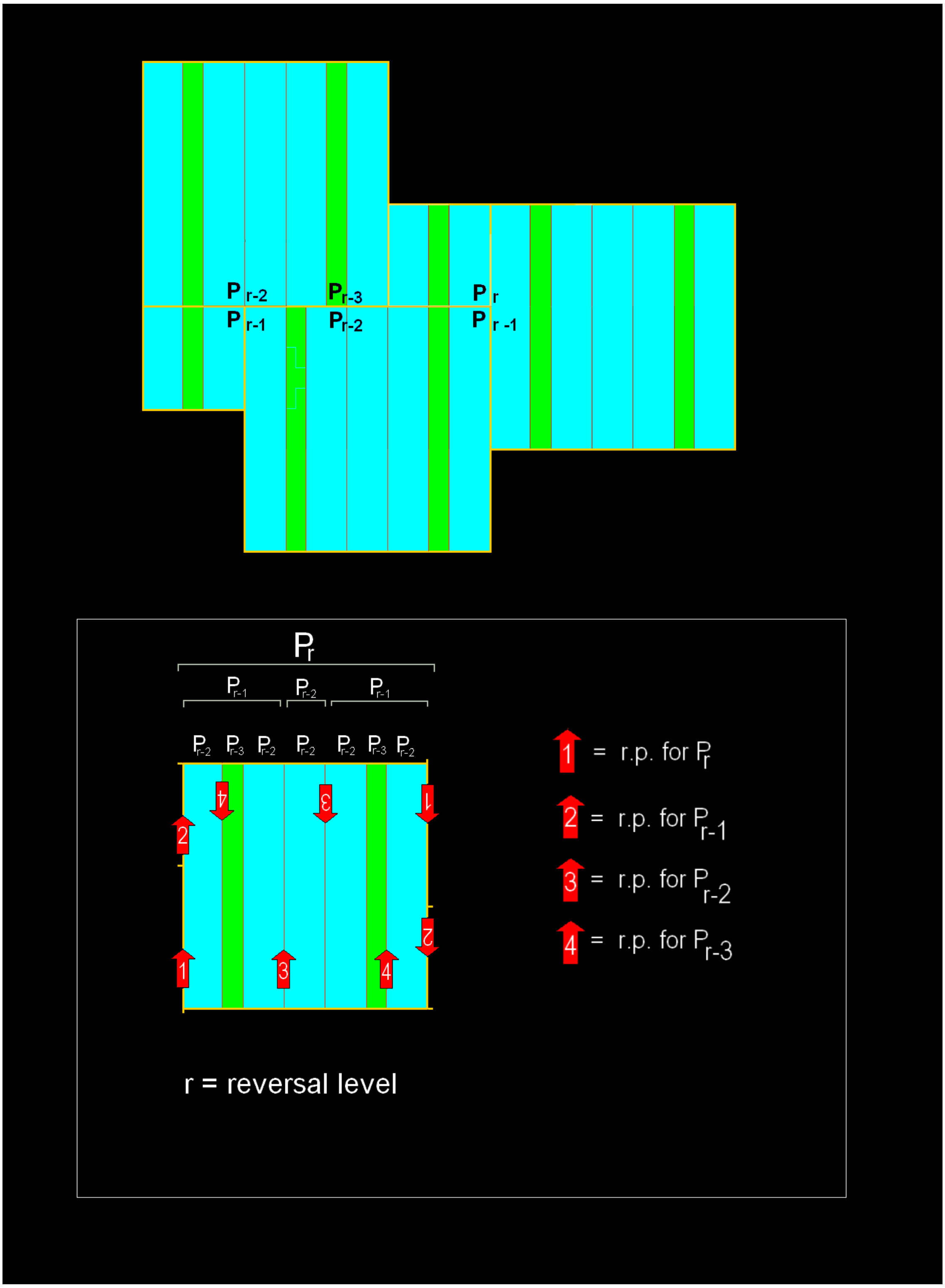


Figure 106*: When the s/l ratio becomes larger, the diversity in the range of the r.p.'s becomes greater, and so of the number of locations at the borderline between large squares where the level of the r.p.s dimishes in the transition of orange lines from one square to the other. In this example s/l = 5/12.*

This square after square further diminishing of the level of the r.p.'s is directly related to the layeredness in the built up of the crystals. But to illustrate this , we first need to make the picture complete and bring both orientations of the rp-lines in picture, instead of only one. Until now, to get grasp on things, we worked with an explicit demarcation of the border lines of the squares. But by demarcating these border lines it difficult to realize a transparent picture of the interaction of rp-lines when we bring in picture both orientations. When we have a closer look at the model in Figure 100 (right picture) and look at the border lines of the squares, we see that these border lines only in one orientation coincide with rp-lines. They all traverse through the middle of pieces that we have indicated as U-pieces. In the other orientation the border lines are not rp-lines. They only demarcate the possible change in the status of rp-lines that run in the other direction, becoming carriers in those cases of higher or lower order r.p.'s. When we work with two orientations of rp-lines, these last mentioned border lines make the picture unnecessary complex and blocks transparancy. Now we have learned the things we have to learn from the one orientation approach, there is no longer reason to

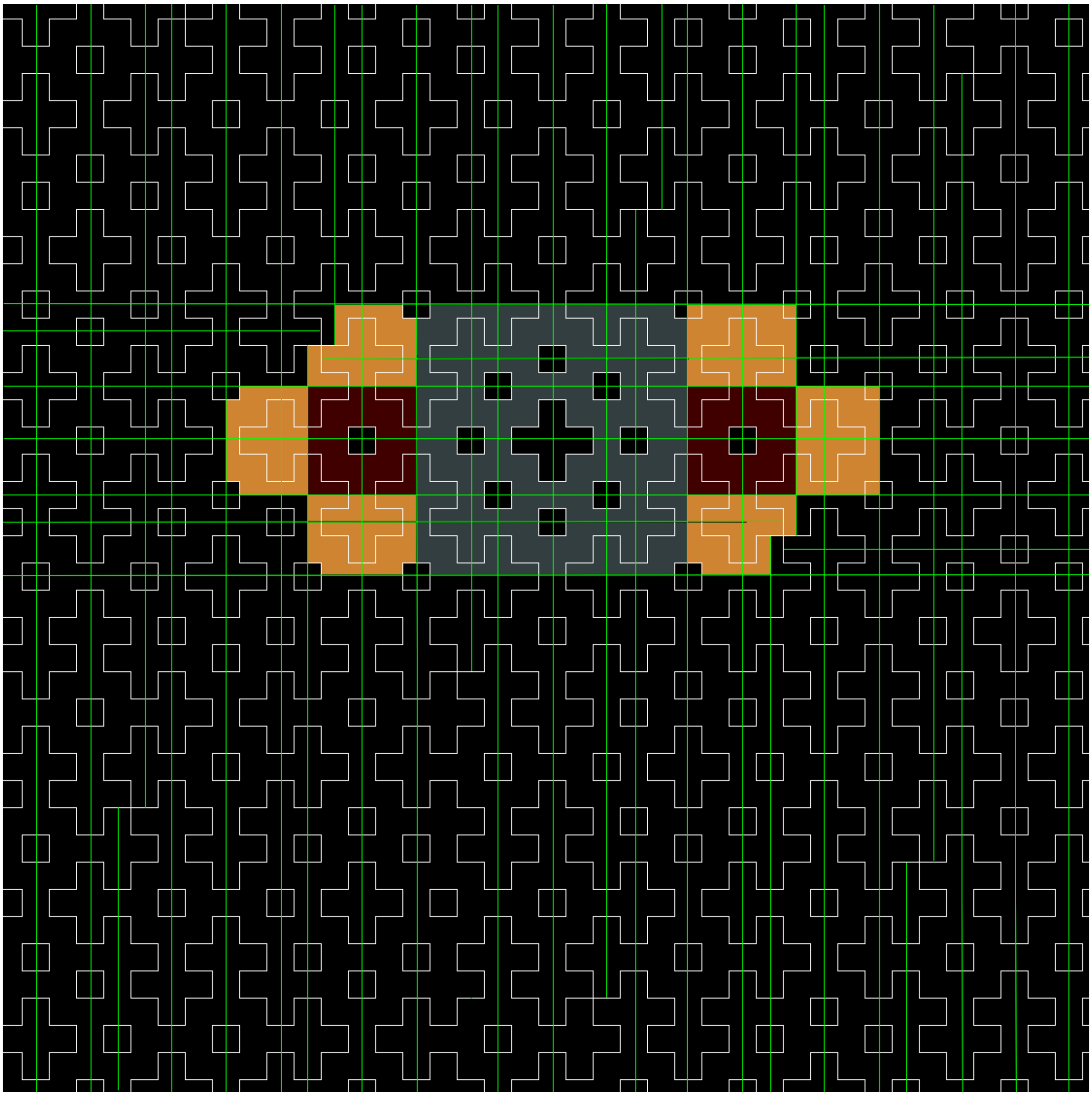


Figure 107*: Indicating the rp-lines in two orientations.*

demarcate the border lines of the squares. So we leave the demarcation behind us. Than than the interaction of rp-lines in two orintations results in wonderful fractal like structures , in which layered crystals quite naturally loom up. Figure 107 shows the example *s/l* = 5/12 , which has as partial quotints [2, 2, 2]. The more layers in the continued fracton of *s/l* , the more layerdness we see in the crystals. Figure 108 shows the example s/l= [2,2,2,2,2].

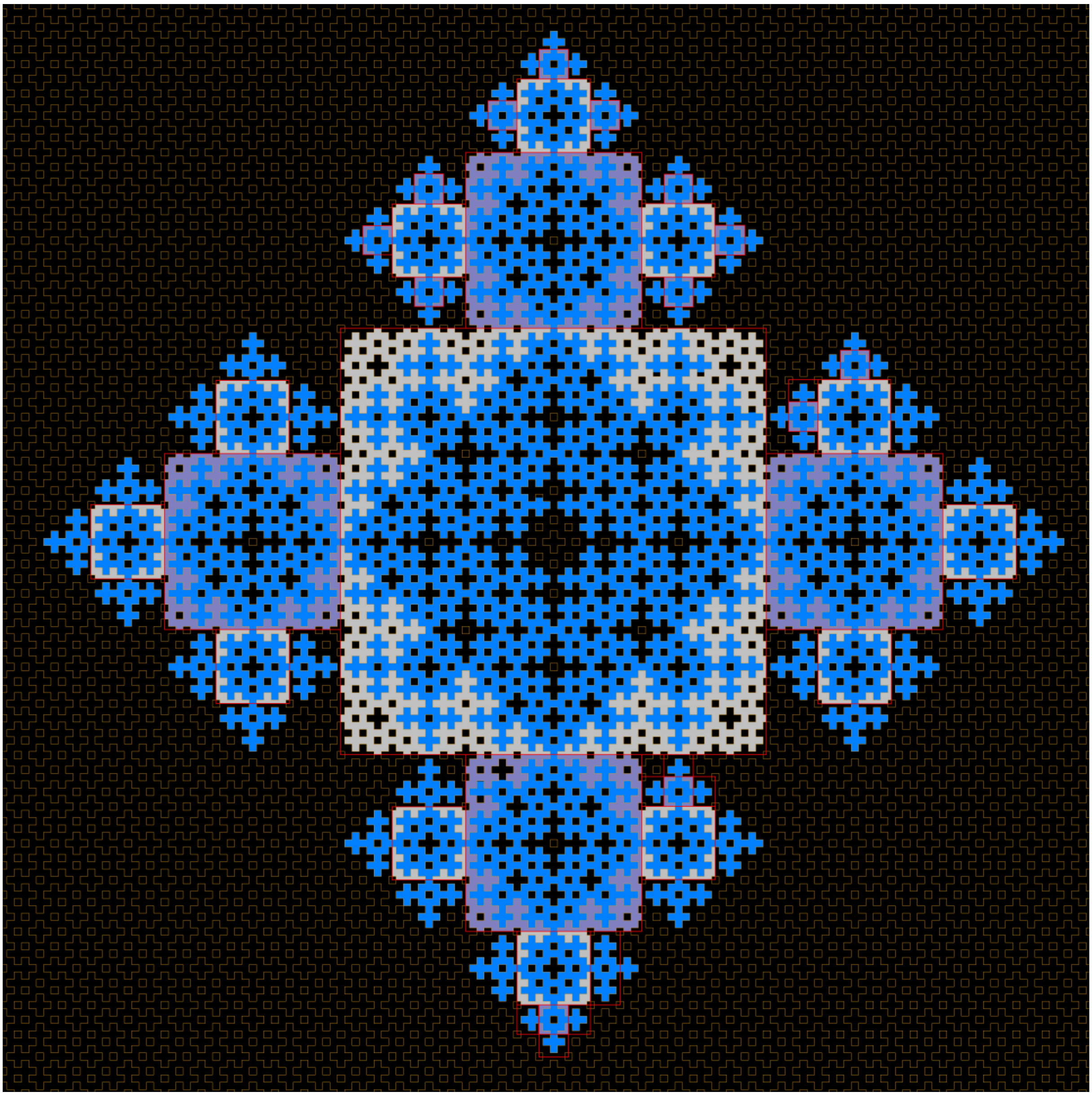
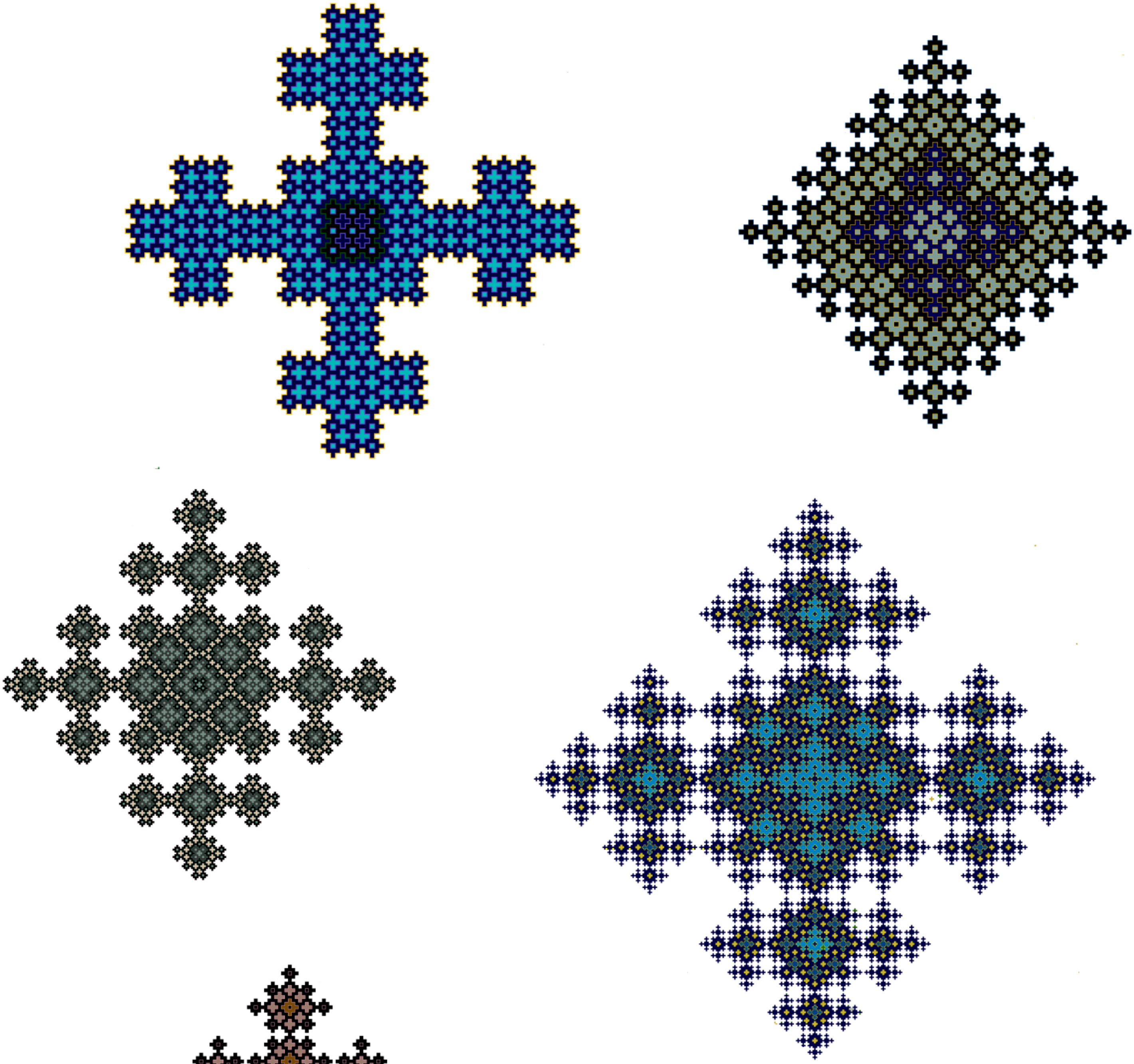
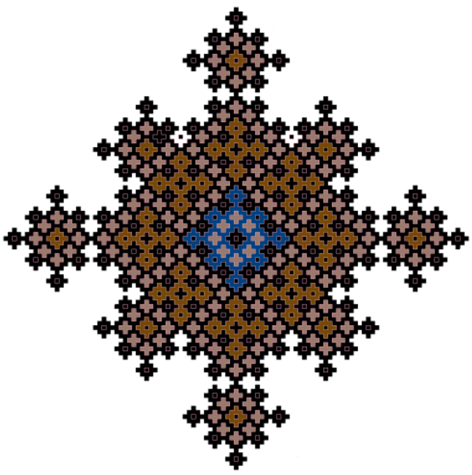


Figure 108: *The more layers in the continued fracton of s/l , the more layerdness we see in the crystals. Figure 108 shows the example s/l= {2,2,2,2,2].*

5.4 beautiful crystals can be realized

Beautiful crystals can be generated by the in this section described approach for generation of Christoffel sequences (Figure 109)





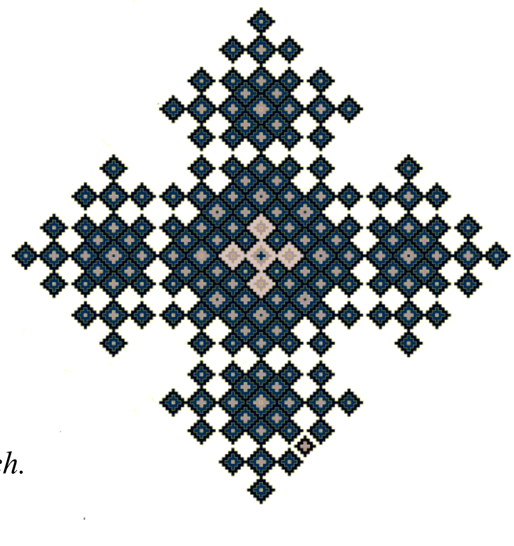
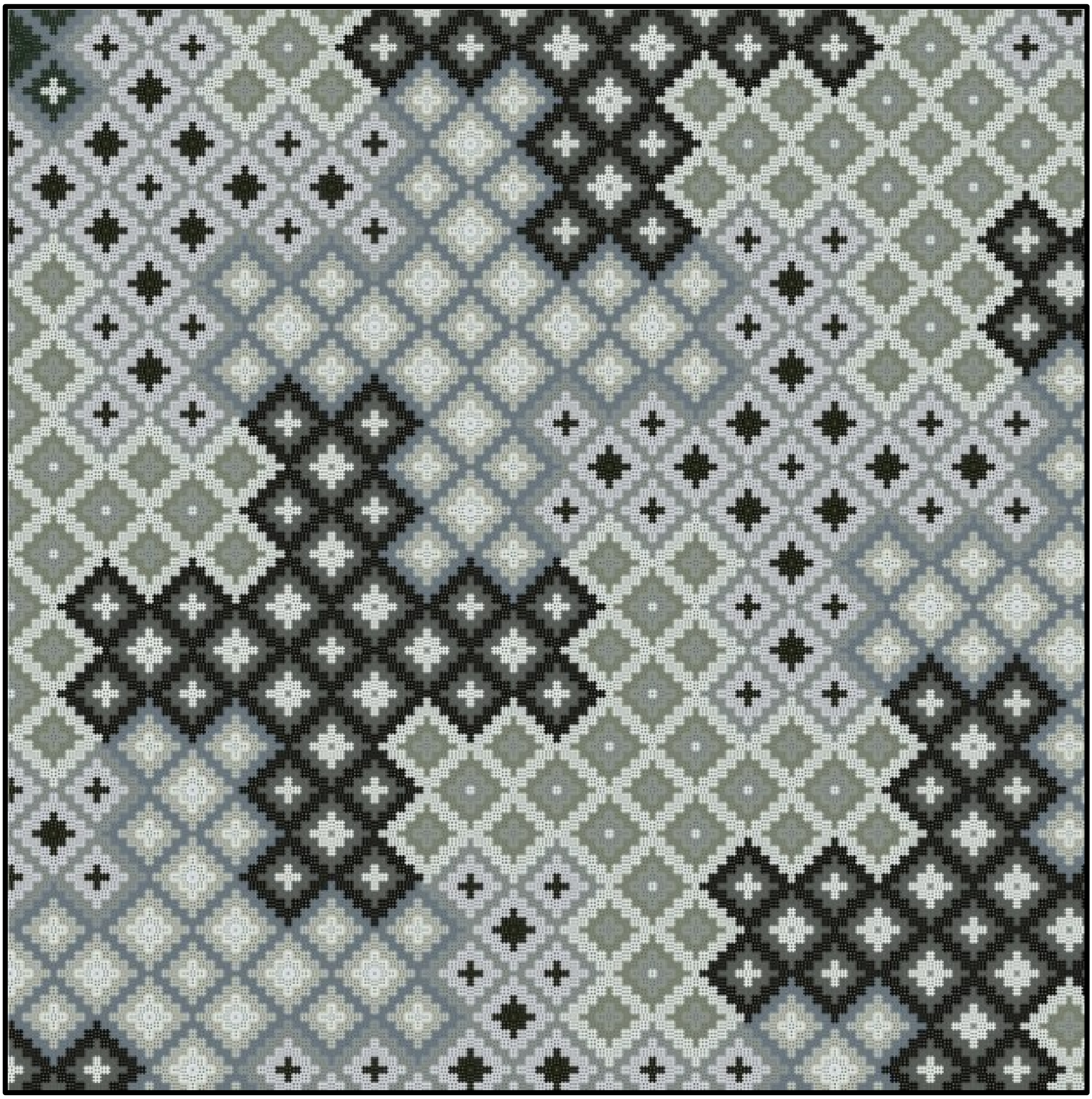


Figure 109: *Beautiful crystals can be generates by this approach.*

Not only the crystals on their own , but also the patterns in which they show up can have high esthetic quality (Figure 107).



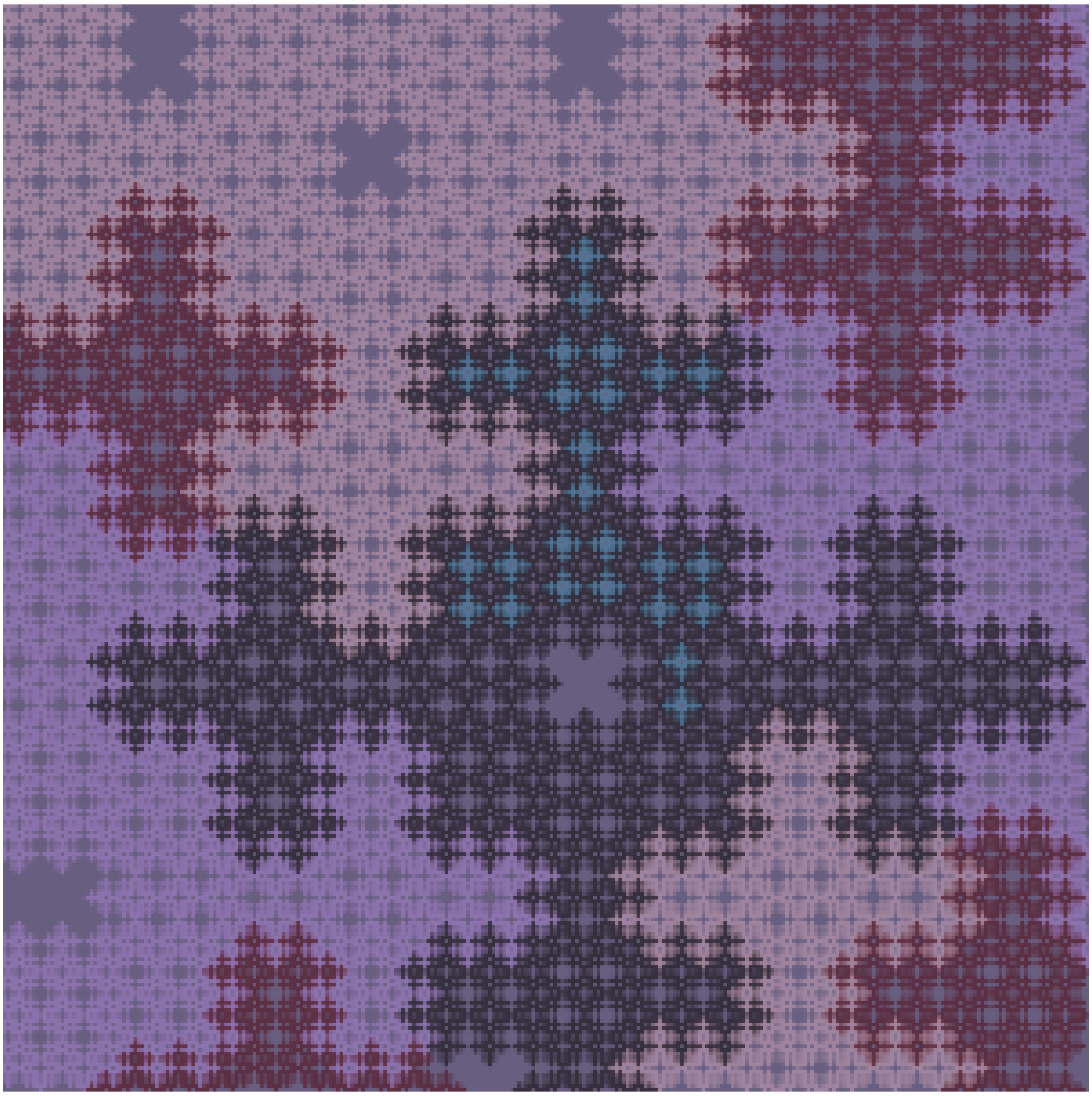


***Figure 107**: Pattern in which crystals show up also can have high esthetic quality.*

Generating such beautiful crystals patterns is not only possible in p4 but also in p6 (Figure 108) . But discussing the pattern generation in p6 is a ' bridge to far' in this paper. he cross

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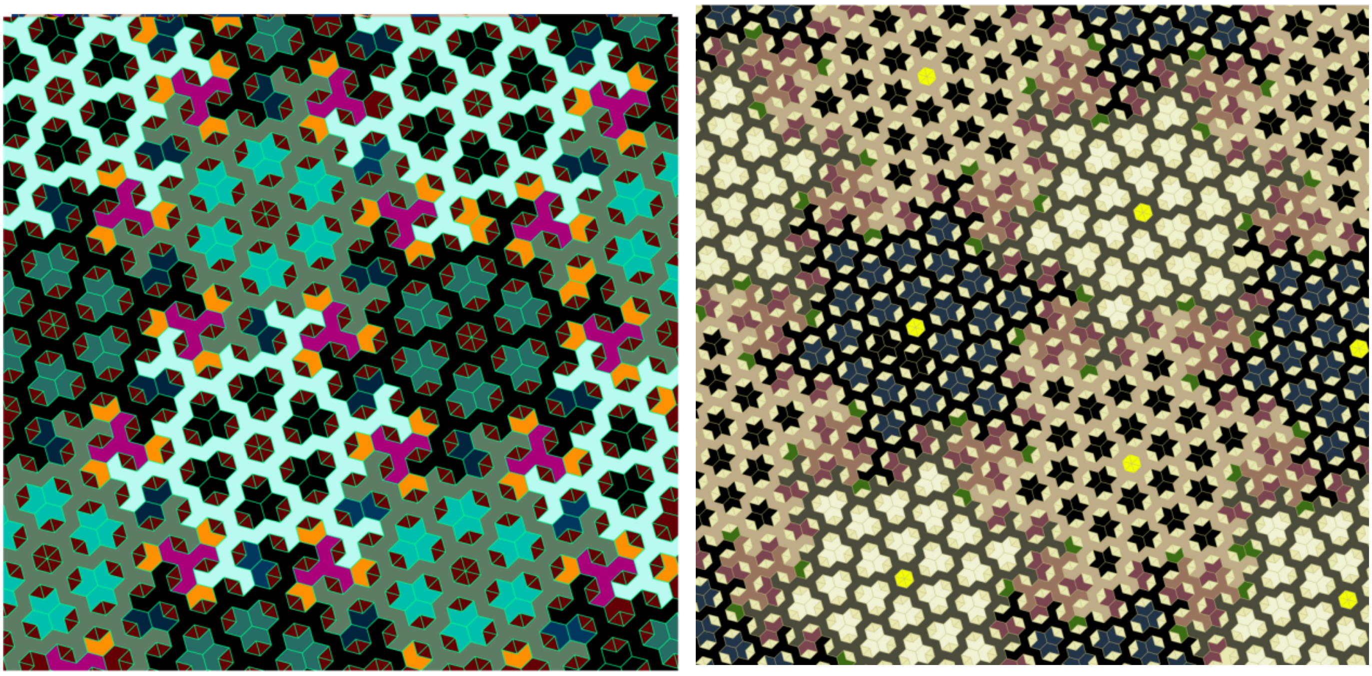

Figure 110 *Patterns generated by introducing direction related Christoffel sequences in p6.*

11 Rainy Sunday afternoons

Several years have passed since, in matters of symmetry and order, I'm able to let bygones be bygones. The obsession which had a hold on my mind during a large part of my life is at long last lying doggo. Nowadays I spend my time with the mountains of publications about Christoffel sequences that was brought forward by the area of mathematic which has become known as 'Combinatorics on Words ' . Mathematician who devoted their life to repetition patterns within sequences of numbers or letters. Immersing myself in their megalomaniac way of reasoning in which they seem to think that reality itself can be grasped in terms of 'free monoids'. Surprising me again and again about that fortification of proved theorems that through the decades was erected by them. Without much concern about the ' why' of that fortification. And then, with a touch of 'Schadenfreude', I note that in there scientific language they never shall understand the beauty of symbols like the cross and the swastika. Symbols which are so closely intertwined with the history of humanity. Because their thinking is to generalistic and their approach to rectilinear. In the urge to be universal in their way of grasping reality, they were disregarding and ignoring the importance of the reversion-point-particles in palindromes. And for that reason they never shall understand the magic and beauty which are inherent to two most deeply with the history of humanity intertwined geometrical shapes, namely that of the cross and the swastika. These mathematicians have so to say lost their paradise, like Adam and Eve, because they were to haughtily. It's their punishment for carelessly spoiling the heritage of on number theory oriented giants like Henry Smithy an Caroline Series who were shedding so much light on those wonderful sequences. A heritage left by giants like Henry Smithy an Caroline Series.

Nowadays my head is still brimming of mathematical reasoning. But very occasionally that overwhelming desire to make beauty out of certain magic rules which are hidden in structures of perfect order, which I know so well from former years, comes to the surface again. Especially on rainy Sunday afternoons , when life seems forever to have taken a solid state in grayness and monotony and is making you feel blue in every fiber of your being. Under those conditions that old passion of mine suddenly may blaze up again in full force. Giving me the assurance that there is such immense beauty inherent in structures of perfect order that many an artist might wish that this beauty had originated in his own mind, instead of having been present in crystal structures from the beginning of time, long before Man came on earth to wonder at the marvel of it.

Finally

To those readers who still in our days feel upset by the appearance of the swastika in a paper like this, I value to say the following words. It is regrettable that the Nazi beasts in the past century have besmirched this symbol so much. It's a symbol deeply rooted in cultural expression of mankind, symbolizing life, sun, power, strength, good luck and eternity.

Figure 73: *Swastikas embracing crosses.*

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